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MATHEMATICAL THEORY OF HUMAN RELATIONS

AN APPROACH TO A MATHEMATICAL BIOLOGY OF SOCIAL PHENOMENA

BY

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PREFACE AND EXPLANATORY REMARKS

Social sciences for a long time were considered, and frequently still are considered, to be definitely distinct from the natural sciences. Yet the fundamental object of study in the social sciences is man, a form of animal, and therefore an object of study of natural science. Human behavior, which is the determining factor in many, if not in all, social relations, is the object of study of psychology—that is, of a natural science. The eminent ecologist W. C. Allee chooses the following subtitle for his book on animal aggregations:[1*] "A Study in General Sociology." With good reason, sociology may be considered as a very special branch of ecology. The latter is defined as the study of the interaction between organisms and their environment. This environment, however, consists to a large extent of other organisms. The object of social sciences is the study of the interaction of individuals or of groups of individuals with other individuals or groups. The latter are part of the environment of the former.

Mathematics has been a most important tool in the older natural sciences—the physical sciences. For the younger biological sciences its use on a larger scale has been introduced only within the last few decades. Mathematics in biology is, however, rapidly establishing itself. A number of books[2] in the field of mathematical biology and the existence of a journal devoted to that field[3] clearly demonstrate the advances made.

If we consider social sciences as part of biological sciences, then it is only natural to introduce mathematical methods into the former. It is also natural, however, to expect serious objections to such an introduction, objections similar to those which were made in the past towards the introduction of mathematics into biology. It may be claimed that the social phenomena are so complex that any mathematical treatment, with its necessary oversimplifications, is and will remain impossible. It also may be claimed that social relations and social concepts are fundamentally of a qualitative nature and therefore are not amenable to mathematical treatment. Let us consider the two objections separately.

There is a great difference between the statement made by mathematicians that all algebraic equations of fifth degree cannot be solved by radicals and the statement that mathematics cannot be successfully applied to social phenomena. The first statement is a necessary consequence of a set of axioms and postulates on which modern algebra is based. It is a statement which can be proved with mathemati-

* In this book all numbered references are given at the end of each chapter.

cal rigor. It is not a mere opinion on the subject, and on *that* subject there are not, and cannot be, two different opinions. An attempt on the part of any person to solve a general fifth degree algebraic equation by radicals would merely indicate a complete ignorance of higher algebra.

The statement that social phenomena (or for that matter, any given phenomena) are not amenable to mathematical treatment is in essence an opinion, based on the fact that hitherto no successful application of mathematics to social phenomena has been made. While no one can deny this fact in the past, it is legitimate to challenge such a pessimistic extrapolation into the future. History of science has seen so many examples of such negativistic opinions that no scientist can take them too seriously. The complexity of social phenomena will tax the ingenuity of the mathematician, but there are no indications whatsoever that above a certain degree of complexity phenomena become refractory to the mathematical method. Mathematics is not a dead science. It discovers more and more new methods to cope with difficult and complex problems.

It may be correctly remarked that the more optimistic opinion regarding future uses of mathematics in social sciences is also nothing but a statement of belief. We concede this point, but we remark that this more optimistic opinion not only is supported by the history of natural sciences, but also leaves the door open for further attempts at new developments instead of closing it forever.

Now let us consider the objection that social phenomena are essentially qualitative in nature and therefore are not amenable to mathematical treatment. Again, we must concede that not only in social sciences, but also in some natural sciences, we sometimes deal with relations of a qualitative nature. Mathematics, however, can frequently be used in such cases also. Such branches of mathematics as Boolean algebra or topology may well be said to deal with qualitative, rather than quantitative, relations. The usefulness of Boolean algebra in the theory of some neuropsychological phenomena has been demonstrated.[4] But in the field of social sciences we observe such varied phenomena as, for example, the distribution of national incomes in different countries, different incidence of crimes, different divorce rates, different kinds of shifts between urban and rural populations, different incidence of inventions, etc. Those are definitely of a quantitative nature. Using statistical methods, we may empirically establish some mathematical relations between various quantities mentioned above. The natural scientist, however, does not remain satisfied with quantitative empirical relations. He attempts to *derive* those relations from a set of quantitatively formulated postulates, and

thus to build a systematic theory which not only enables him to describe correctly already known quantitative relations, but also to *predict* new relations which have not yet been observed.

Thus an objection to the use of mathematical methods in social sciences based on the allegedly qualitative nature of social phenomena would indicate both a lack of knowledge of more qualitative branches of mathematics and a disregard for a large array of quantitative sociological facts.

There remains, of course, the possibility for some students of social phenomena to say merely that they are not interested in any "mathematical theorizing" and that such theorizing does not even constitute social science. Nothing can be said against such an attitude which amounts to a narrower *definition* of the domain of social sciences. Like any other definition, such a definition is perfectly legitimate, although also purely conventional.

Quantitative studies are not new in social sciences. They are found to some extent in a very large number of books on sociology. Of all such studies one should receive special mention in connection with the present work. We refer to P. Sorokin's[5] "Social and Cultural Dynamics," which is particularly permeated with the quantitative spirit characteristic of natural sciences. It has been objected that the data used by Sorokin are not always reliable and that therefore his quantitative conclusions are of doubtful value. Such an objection, however, misses the point entirely. In the early development of physics, the data used were also crude and frequently unreliable. But they gave some crude quantitative idea about different phenomena. The law of Dulong and Petit, stating that the molar specific heat of solids is a constant, is an example of such a crude set of data. We know now that it holds in a very limited range of temperature, and at that only as a rough approximation. No one would use that "law" now when we have the much more exact equation of Debye. But no one will also deny the useful role that law played in physics.

What has been hitherto lacking in social sciences is a systematic development of a mathematical theory which would enable us to derive various known quantitative relations. S. C. Haret's[6] "Mécanique Sociale" has been doomed to failure, in our opinion, because of a too literal imitation of physics. The tremendous achievements which physics has made through the use of mathematical methods indicates the usefulness of following the same *method* in other sciences—in particular, in sociology. But the imitation of the *method* does not mean a literal imitation of form. While, as we shall attempt to show in this book, there is every reason to justify the use of mathematics in sociology, there is no reason whatsoever to justify the assumption that the mathe-

matical equations which govern social phenomena are of the same form as the equations of Newtonian mechanics, as Haret postulates. Even in different fields of natural sciences entirely different branches of mathematics are used. Haret imitates the physical sciences in the letter rather than in spirit.

In the present volume the author attempts to outline a system of mathematical sociology in the spirit of mathematical physics and mathematical biology. As we have seen above, objections to such an attempt cannot be considered valid. But any statement in favor of such an attempt must remain just as invalid until the success of the attempt is proved by actual demonstration. An attempt at such a demonstration is made here. The task which the author undertakes is a difficult one, and he is well aware of the fact that others could do the same thing in a much better way. Someone, however, has to start a new field despite the fact that the first attempt is likely to be full of shortcomings, as is actually the case with the present book.

To facilitate critical reading and better understanding of the book, let us briefly review the most important shortcomings.

A mathematician is likely to find the mathematics used here too elementary to arouse his interest. On the other hand, a person with only a reading knowledge of mathematics is apt to feel the presentation is too short and therefore difficult to follow. Striking the golden middle, if such a golden middle exists, is a difficult task, and the author is ready to concede that he has not achieved it successfully.

It is a bad policy to begin a book with relatively complex and difficult mathematics and continue it with the use of much simpler methods. However, this is what the author is doing. The first two chapters involve some knowledge of integral equations, a knowledge which many readers may not possess although they may be well equipped to understand most of the relatively simple mathematics throughout the rest of the book.

In the first chapter the author establishes some general mathematical relations which are likely to describe the structure and behavior of a society. Those relations are sometimes in the form of integral equations. Since the actual solution of such equations is difficult and at times impractical, the author resorts in chapter iii to a drastic approximation which reduces the problems to ordinary differential equations. Logically, the presentation of the general problem at the beginning is justified. Didactically, however, it might have been better to present the approximate treatment first and then at the end to formulate the problem in all its generality. One excuse for following the method accepted by the author is that although the notion of integral equations is used as early as page 5, nevertheless no actual

solution of any integral equation is attempted in either of the first two chapters. Moreover, the discussions on pages 3 and 4, which require only elementary knowledge of integral calculus, lead rather naturally to the discussions on the following pages, and any reader with the knowledge of integral calculus is not likely to experience any difficulty. It may be added that a less mathematical reader may skip the first two chapters if necessary.

The various relations established in chapter i give a general mathematical description of some social phenomena, such as distribution of some activity among individuals of a society, formation of social groups, spatial distribution of individuals, and occurrence of possible instabilities of a society. These equations are, however, of little practical value, partly because of the generality of their formulation, partly because of the mathematical difficulties involved in their solution. Chapter ii is essentially a continuation of chapter i. An attempt is made here to illustrate how some sociological quantities which cannot be measured directly can in principle be indirectly computed. But again the discussion remains on a rather abstract level. The rather abrupt ending of both chapters i and ii also is a decided shortcoming in the presentation.

In chapter iii the author introduces an approximate method which at once avoids the use of integral equations and even of more simple integral expressions. This approximate method is used throughout most of the book. Although the author emphasizes that a sharp division of a society into "active" and "passive" individuals is only an artefact, used for mathematical expediency, the reader may perhaps gain the impression that the assumption of such sharply defined classes is an essential sociological postulate used by the author. That this is not so is demonstrated in chapter iv, where it is shown that results essentially similar to those of chapter iii are obtained by considering a much more realistic case, in which various characteristics of individuals vary continuously and where no sharp distinction between classes exists. The discussion of chapter iv does not, however, dispose of all the objections which may be raised to chapter iii. First of all, only one of the cases treated by the approximate method in chapter iii, namely, the case of constant coefficients of influence, is generalized in chapter iv. Second, the introduction of the threshold quantity Δ (page 36) leaves the generalization rather unsatisfactory. The elimination of the quantity Δ and the extension of the method of chapter iv to other cases treated in chapter iii is definitely needed. A further serious shortcoming of chapter iii is the limitation of the treatment to only two active classes. An extension to any number should be made. The author is happy to state that between the time when the

manuscript of the book went to the printer a year ago and the time of the writing of this preface, a large portion of the above-outlined program has actually been accomplished and is awaiting publication in its turn.

Returning again to chapter iii, legitimate objections may be raised to the rather simple assumption of linearity which leads to equations (2) and (3). It may be more natural to assume that the frequency of interaction between active and passive individuals is proportional to the product of their respective numbers. This would lead to non-linear equations, with possibly very different results. The only justification for the assumption of linearity is its simplicity—an argument frequently honored in mathematical sciences. Further studies must, of course, include possible non-linear cases.

Another shortcoming of chapter iii should also be mentioned. The reader may gain the impression that the behavior of a social group will change only if x_0 or y_0 changes. This, of course, is not so, because inequalities (8) and (9) may be reversed by varying the coefficients a_0, c_0, and a. Although most readers will see this, a more explicit statement to that effect should have been made in the text.

As pointed out at the beginning of chapter v, the reader is likely to be puzzled by the crude economic concepts introduced there, in spite of the existence of much more developed mathematical theories of economics. This, as explained, is due to the desire to emphasize the social aspects rather than the purely economic ones. The chapter suggests a mathematical description of some simple socioeconomic relations. The author, however, is fully aware that it is probably the weakest chapter in the whole book.

The first section of chapter vi deals with an extension of some considerations of chapter v, and suffers essentially from the same defects. The second section introduces a new approach, which is also used later in chapter xix. The main objection to that part of the chapter is its sketchiness and lack of further elaboration.

At the end of chapter vi the author gives what may be considered the first "application" of the theory. After outlining a general program for a mathematical study of some economic relations, he derives an approximate expression for the per capita incomes of different countries and compares the calculated values with available data. Many readers will undoubtedly be appalled by the crudeness of the derivation. Although the author points out that equation (43) is derived only as an illustration, this should be emphasized much more strongly.

From a set of postulates we may develop a theory and then compare the quantitative consequences of the theory with observations in

order to see whether our postulates lead to correct predictions. In such a case the derivation must be sufficiently rigorous and the observations reliable. However, if we have a field in which such a procedure has not been used before, we may wish, even before developing a sufficiently complete theory, to give an example of what kind of relations a theory may predict and what sort of data may be needed for comparison with the theory. It is with such a purpose in mind that the author derives equation (43) of chapter vi and some other equations in subsequent chapters. The crudeness of the derivation is then somewhat justified. In all such cases the author is also fully aware of the inadequacy of the data he used, and he did not make any special effort to select critically the best data available. These remarks also must be kept in mind when reading chapters vii, x, and xvii, where other expressions are derived and compared with observations only for the purpose of *illustrating* the general method.

The presentation of chapter viii is somewhat too condensed. It might have been wiser to explain in detail the reason for studying situations in which the effort of individuals to achieve a certain goal decreases with *cumulative* success in the past. Psychologically, however, such a situation is rather plausible. The same objections which were raised with regard to chapter iii hold for this chapter.

Chapter ix outlines a possible quantitative treatment of the concept of individual freedom. Two cases are discussed, as an illustration only. Many other possibilities, of course, are present. The general conclusion that individual freedom, as defined in that chapter, decreases with increasing number of individuals seems to lead, as pointed out to the author by Dr. A. Rapoport, to a peculiar paradox. An individual confined to a solitary cell in jail is deprived of his freedom and yet is also removed from contact with other individuals. It must be remembered, however, that no individual lands in jail without first coming into some kind of conflict with other individuals.

Much of what has been said about chapter vi can be said about chapter x. The assumption expressed by equation (4) certainly does not correspond to actual facts. This objection may be overcome by considering, instead of per capita production, per capita satisfaction, as discussed in chapter vi. An individual will then move either to the city or to the country, depending on where his satisfaction will be greater. The satisfaction s is, in general, a function not only of p_u or p_r, but also of other factors which will enter as parameters λ_i. Instead of equation (4) we then shall have

$$s_u(p_u, \lambda_i) = s_r(p_r, \lambda_i),$$

where s_u and s_r are different prescribed functions. The same remarks

may be made about chapter xi.

In chapter xii an attempt is made to derive the distribution of city sizes on a different basis. The derivations are rather involved, and the results none too definite. The approach could stand a great deal of improvement.

Chapter xiii is decidedly too sketchy. It may, perhaps, create the impression that the mechanism of history suggested there necessarily follows from any general mathematical theory. The author emphasizes, however, that this is not so, and that other mechanisms may be conceived.

Chapters xv and xvi suffer from diametrically opposite defects. In the former, the mathematical treatment is too simple and hardly corresponds to reality. In the latter it is too involved and suffers from lack of elegance. Instead of studying several special cases, a general solution should have been given, if possible.

All that has been said above concerning section II of chapter vi must be said again about chapter xvii. All examples treated there are to be considered *as illustrations only*. In the subsequent development of the theory, the general method used in chapter xvii could be applied, however. Instead of using relation (1), which does not correspond to facts, the relative values of x_0 and y_0 could be computed from a comparison with appropriate observations of different relations which contain those values.

Chapters xix and xx lead to some interesting consequences. However, as the author emphasizes there, those consequences are obtained *only* with the particular set of assumptions made. A change in any of those assumptions may change the results. If, after reading chapters xix and xx, anyone "jumps at conclusions," he does so at his own risk!

The concept of satisfaction function seems to be very useful in the mathematical theory of social behavior. Following the approach suggested in chapter xix, Anatol Rapoport has obtained very interesting results in the theory of motivational interaction of two individuals.[7] In a still unpublished paper Anatol Rapoport and Alfonso Shimbel show that animal behavior can also be described in terms of a satisfaction function, the form of which could be determined by properly designed experiments.

The discussion of chapter xxi is again intended primarily as an illustration of a possible quantitative approach to history. The suggested assumption of a connection between the original distance of migration of a population and the number of active individuals in it cannot, of course, be accepted in this simple form. From such an as-

sumption we would expect the largest percentage of active individuals among such peoples as Eskimos or American Indians. There is a possibility, however, that while the assumed connection exists, other factors not yet known upset it in some cases.

The last two chapters deal with the mechanism of physical conflict between social groups, and contain an outline of a possible mathematical theory of war. In its present form the theory is oversimplified. Further mathematical developments, however, should enable us to treat many aspects of war, using available material such as, for instance, Quincy Wright's "A Study of War."[5]

After reading all the self indictment above, anyone may legitimately wonder why the book has been published at all. Let us look at some positive sides of it.

It is shown how the structure of a society and the behavior of its members can be mathematically described in terms of integral equations. It is shown how social groups or classes are formed within a society and equations are derived which determine the size and number of such groups. It is shown how the interaction of these groups determines the behavior of a society as a whole, and it is shown that this behavior may change either gradually or suddenly, in a revolution-like manner. Equations are derived for the time intervals between such sudden changes and for the speed of the changes, and the calculated values are found to be of the same order of magnitude as some observed ones. Mathematical relations are derived which govern the physical conflict between social groups. The outcome of such conflicts is shown to depend not only on purely physical factors, but also on such psychological factors as morale. A theory of "breakdown of morale" is suggested.

It is shown how mathematical relations could be derived for such diverse sociological quantities as national incomes, sizes of cities, ratio of urban to rural population, crime incidence, divorce rates, war expenditures, and cultural productivity. Each case is illustrated by actual examples.

It is for the reader to decide to what extent all this compensates for the shortcomings of the book.

The author wishes to express his thanks to all those who helped him in the preparation of this work. The largest part of the manuscript has been typed by the author's wife. Some chapters were prepared by Mrs. W. Maertz. To Dr. Anatol Rapoport the author is indebted for a very critical discussion of the final manuscript and for the checking of all calculations. Professor Harold T. Davis has also read the manuscript and made valuable comments. Mrs. J. Rall has had the important but unrewarding task of painstakingly careful

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INTRODUCTION

In the absence of any other individuals, the behavior of an individual would be determined by his immediate surroundings, his psychophysical constitution, and his past history. Mathematical Biophysics has shown that a physicochemical interpretation of many rather complex psychological phenomena is possible.[1] If *the* real interpretation were actually known, those purely psychological quantities would be amenable to quantitative expressions in terms of the intensities of the corresponding physicochemical processes in the organism, and natural units for their measurements would thus be established. Our very limited knowledge of those underlying physicochemical processes makes such a thing still impossible *at present.* But inasmuch as we admit a physicochemical interpretation of psychology, we should not hesitate to introduce such things as "desire" or "will" as quantities in our equations and consider them *measurable in principle,* although they may not be measurable *directly.* As a matter of fact, such an introduction of those quantities into our fundamental equations may ultimately result in the possibility of their *indirect* measurement. In developing mathematically the consequences of those equations, we may arrive at expressions which connect those quantities with directly measurable ones. Such a procedure is not unfamiliar to physics and, in our opinion, those psychologists who declare war on anything which is not directly measurable and invoke as a justification of this war the example of physics, are "plus royalistes que le roi."

In classical electrodynamics such fundamental things as the electric and magnetic vectors are not *directly* measurable quantities. They are measured by *definition* by the force which is exerted on a unit of charge, and the force is also measured indirectly by measuring displacements of a dial on a scale or by observing accelerations. Of course, the *indirectness* of the measurements here is *relatively* small, but this does not affect our argument in principle. Some may quite legitimately prefer to define the electric and magnetic vectors as quantities that satisfy Maxwell's equations. The indirectness of their measurements then becomes still more evident. Physics has been attempting to eliminate quantities which are not *measurable in principle.* But there is a great difference between "measurable in principle" and "measurable directly." In considering the prominent role

of Schrödinger's ψ-function and its generalization by Dirac, we may still question whether the complete elimination even of *quantities not measurable in principle* will be the ultimate goal of physics.

As regards the comparative value of direct and indirect measurements, witness the fact that calculations of the values of e and h (electronic charge and quantum constant) from spectroscopical data lead to much more accurate values than direct measurements.

Inasmuch as *at present* we are dealing with purely abstract constructions, the introduction of some quantities not directly measureable is justified *a fortiori*.

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CHAPTER I

GENERAL MATHEMATICAL CONSIDERATIONS

I

Denote the intensity of some activity i by a_i. For a single isolated individual, a_i will be proportional to the desire w_i for this activity, which is, in its turn, determined in general by the past history of the individual. If another individual is present, he will in general influence the activity of the first one. This influence will be determined by two factors: the desire w'_i of the second individual for a_i, and the amount of influence which the second individual exerts on the first. This influence may be of different nature; it may reside in physical force (which happens to be an *almost directly measurable* quantity!) or in any other kind of influence, such as moral, religious, etc. Considering first the simplest possible law of interaction, a *linear* one, we shall put the influence of the second individual upon the activity of the first one as proportional to a coefficient of influence $F' > 0$ of the second individual, and to w'_i. Hence we have

$$a_i = \alpha w_i + \beta F' w'_i , \tag{1}$$

α and β being constants; w_i and w'_i may be either positive or negative. If $w_i > 0$, $w'_i < 0$, the second individual may be said "not to like i," and attempts to reduce the activity of the first.

Let us now consider a large group of $\mathfrak{N}$ interacting individuals, and first suppose that there is only one kind of activity a which they all perform. We can characterize each individual in this case by his F and w. In general all individuals differ from each other. There will be a distribution function

$$N(F, w)\, dF\, dw , \tag{2}$$

which indicates how many individuals are characterized by values of F and w lying in the intervals F, $F + dF$ and w, $w + dw$. The distribution function (2) satisfies the relation:

$$\int_0^{\infty} \int_{\infty}^{+\infty} N(F, w)\, dF\, dw = \mathfrak{N} . \tag{3}$$

Every individual (F, w) performs the activity with the intensity

$a(w)$ which is determined by his own w and by the influence of all others. Accordingly, equation (1) now becomes:

$$a(w) = \alpha w + \beta \int_0^{\infty} \int_{-\infty}^{+\infty} N(F', w')\, F'w'\, dF'\, dw'. \tag{4}$$

In the most general case we may consider the function $N(F, w)$ as possessing a finite number of discontinuities. In general we shall have an integral and an ordinary sum, the latter extending over single individuals who, due to their very large values of F, may exert a large influence on the activities of others.

Equation (4) determines the activity of each individual in terms of the influence of the rest of the community upon him.

We may consider a more general case of interaction which is closer to real conditions. We may assume the influence of an individual (F', w') on (F, w) to depend on the difference $F' - F$. If there is an influence only when $F' > F$, then we have

$$a(F, w) = \alpha w + \beta \int_F^{\infty} \int_0^{\infty} N(F', w')\, (F' - F)\, w'\, dF'\, dw'. \tag{5}$$

In the more general case, when there is an influence even for $F' < F$, we may put:

$$a(F, w) = \alpha w + \beta \int_0^{\infty} \int_{-\infty}^{+\infty} N(F', w')\, (F' - F)\, w'\, dF'\, dw'. \tag{6}$$

A still more general case to be considered next is where the activity of an individual is not only affected by his own desire for it and the desire of others, but where it is also affected by the activities of the others. An individual may *imitate* another one, and if this other individual engages in an activity, i', with an intensity a', then the intensity a of the activity of the first individual will be the stronger, the stronger the a'. We have

$$a = \alpha w + \beta |F' - F|\, w' + \gamma I a', \tag{7}$$

where I may be called the *coefficient of imitation* and γ is a constant. It is to be considered as a third characteristic of an individual, together with F and w. We now have for the whole group of $\mathfrak{N}$ individuals:

$$\begin{aligned} a(F, I, w) = \alpha w &+ \beta \int_0^{\infty} N(F', I', w')\, |F' - F|\, w'\, dF'\, dI'\, dw' \\ &+ \gamma I \int_0^{\infty} N(F', I', w')\, a(F', I', w')\, dF'\, dI'\, dw'. \end{aligned} \tag{8}$$

The individual may imitate the activity of another one in proportion to the influence F' of that other. We then have:

$$
\begin{aligned}
a(F, I, w) = \alpha w + \beta \int_0^{\infty} N(F', I', w') \, |F' - F| \, w' \, dF' \, dI' \, dw' \\
+ \gamma I \int_0^{\infty} N(F', I', w') \, F' a(F', I', w') \, dF' \, dI' \, dw'.
\end{aligned}
\tag{9}
$$

For a given $N(F, I, w)$, the first integrals of the right-hand sides of equations (8) and (9) are known constants, and expressions (8) and (9) are integral equations for the determination of $a(F, I, w)$, with respective degenerate kernels:

$$
K(F, I, w, F', I', w') = IN(F', I', w');
$$

and

$$
K(F, I, w, F', I', w') = IF'N(F', I', w'). \tag{10}
$$

Equations of this type will hold in a more general case of several activities a_i. As an illustration we may consider the case of two different activities under the following assumptions: the activity a_1 of an individual is suppressed or opposed by the other $(w'_1 < 0)$, unless the first also does a different activity a_2, which the second likes $(w'_2 > 0)$ and for the sake of which he is willing to let the first do some a_1. We then have:

$$
\begin{aligned}
a_1 &= \alpha w_1 + \beta(1 - a_2) \int |F' - F| \, N w'_1 \, d\tau, \\
a_2 &= \alpha_1 w_2 + \beta_1 a_1 \int |F' - F| \, N w'_2 \, d\tau,
\end{aligned}
\tag{11}
$$

where the explicit indication of the variables is omitted for brevity. The quantity N is now a function of F, I, w_1, and w_2; and $d\tau$ stands for $dF' dI' dw'_1 dw'_2$.

Similar expressions may be obtained for more general cases.

We may consider a case in which there is a relation between F, I and w for every individual, due to his biophysical constitution. Then we can reduce the number of variables in the above equations. If, for instance, $I = f(F)$ and $w = f_1(F)$, then F remains as the only independent variable.

In general F will consist of several components:

$$
F = F_1 + F_2 + \cdots + F_i, \tag{12}
$$

such as physical force, moral persuasion, religious influence, intellect, etc. We may consider a great variety of cases where the influence of the different factors is different. Instead of (1) we may have:

$$
a = \alpha w_i + \beta \sum_i \eta_i (F'_i - F_i) \, w', \tag{13}
$$

with the corresponding equation for the whole group:

$$a = \alpha w + \beta \int N \sum_i \eta_i (F' - F) \, w' \, dF' \, dw'. \tag{14}$$

The numbers η_i show the relative importance of the various components F_i of the total F.

We may also consider the more complex case where w consists of two components: one component u represents the desire of an individual to do a himself; the other component v represents the desire to have others do a. We may have $u < 0$, $v > 0$, which means that the individual likes to see the others do a, but does not like to do it himself. We then have:

$$a = \alpha u + \beta_1 v + \gamma_1 \int (F' - F) N(F', u', v') v' \, dF' \, du' \, dv', \tag{15}$$

and similar equations for more complex cases.

Any of the above equations may be made the starting point of an abstract theory of human relations. Equations (4), (5), (6), (8), (9), (11), (14) and (15) all have this in common, that the activity of every individual is dependent on the psychophysical constants of all other individuals. In equations (8) and (9) it also depends on the actual *activities* of those other individuals. The activity of every individual depends furthermore on $N(F, w)$, which characterizes the biological constitution of the whole social group. In further developments we may study the more complex cases of interaction of several groups characterized by different functions of $N(F, w)$. In this way we may study mathematically any possible influence of racial and ethnographic factors. For a given $N(F, w)$ our equations determine completely to what degree the activity of any individual is influenced by others. In particular, we may investigate under what conditions the activity of a minor group will remain very little influenced, but will itself influence the activities of the remaining major group. Usually the independence of the activity of an individual increases with F. But in the case represented by equation (14) this is not necessarily so. A large total F does not yet mean a large independence for any activity. The influence of the various components F_i must be studied.

Without going into details here, we see that we can describe in quantitative mathematical terms what would ordinarily be called social structures and relations. Considering some of the activities as more directly measurable by statistical methods, we may try to derive equations which express such quantities as F, I, w, in terms of these measurable quantities.

It may not seem at first to be much worth the trouble to find mathematically that individuals with a large F have their activity very little influenced by those with a small F. It is fairly obvious, one would say. But the value of the mathematical investigation lies not at all in the statement of this qualitative relation. That can be done without any mathematics. The value of the above equations lies in the circumstance that they give definite analytical relations between F and a, for instance. And, as we have already emphasized, it is only through such relations that we find the possibility of observing and measuring some of the more elusive and not directly accessible quantities.

II

Let us now study the association of individuals into groups or classes. In general the tendency for two individuals to associate will be the greater, the more similar the individuals. We may, however, also consider cases of association between individuals with greatly varying characteristics. For the present we shall restrict ourselves to the former case.

The simplest possibility is that two individuals associate only when the absolute value of the difference of their characteristics is less than a certain amount. For the case of one variable, F, we shall have as the condition of association:

$$(F' - F)^2 < \Delta^2 ; \ (F' - F)^2 - \Delta^2 < 0 . \tag{16}$$

Consider the formation of an association of individuals including those having the highest value F_m of F. How far down the scale of F's will individuals still be included in that association? The answer may be given by the requirement that the sum total of $(F' - F)^2 - \Delta^2$ for the whole group would remain negative. If F_r is the lower limit of F still admissible into the association, then:

$$\int_{F_r}^{F_m} \int_{F_r}^{F_m} [(F' - F)^2 - \Delta^2] \, N(F) \, N(F') dF \, dF' < 0 . \tag{17}$$

The left-hand side of expression (17) is a function $P(F_r)$ of F_r; F_r then is the root of the equation:

$$P(F_r) = 0 . \tag{18}$$

Individuals with $F < F_r$ are not admitted into the "social class" of the larger F's. Similar relations can be established for other variables characterizing an individual.

An expression similar to (17), but in which the integration is

carried out between the limits F_x and $F_{x'}$, determines the size of the next class in which the maximum value of F is F_x, and the minimum is $F_{x'}$. Both $N(F)$ and Δ determine the total number of "classes" in a given society. If $N(F)$ is known, we can calculate Δ from the (usually small) number of "classes."

Other criteria for association into social classes can also be studied. By analogy with expression (13) we may consider as a criterion of association:

$$\sum \zeta_i (F_i - F'_i)^2 < \Delta^2. \tag{19}$$

Finally we may take as criterion not the differences in characteristics but differences in activities. We then have:

$$(a' - a)^2 < \Delta^2, \tag{20}$$

a being determined as a function of F and of the other variables by the fundamental equations chosen to represent a given social interaction. We can determine the number of individuals $N^*(a)\,da$ who have an a between a and $a + da$. Then we have for the determination of the size of the class:

$$\int_{a_x}^{a_m} \int_{a_x}^{a_m} [(a' - a)^2 - \Delta^2]\, N^*(a)\, N^*(a')\, da\, da' = 0. \tag{21}$$

III

Thus far we have considered the interaction of the individuals as being independent of their distribution in space. The next step is to consider the more general case where the location of an individual affects his interaction with others as well as his activity. In this way we can study mathematically the influence of various geographic factors.

First of all, the desire for a certain activity may depend on the location of the individual, so that now, considering only F and w as independent variables characteristic of the individual, we have $w = w(x, y)$, x and y being the geographic coordinates. The interaction of two individuals will also, in general, be a function of their mutual distance $f(x' - x, y' - y)$. As a rule it will decrease with increasing distance. We must also consider a more general distribution function, $N(F, w, x, y)$, which not only gives the distribution function with respect to F and w, but also indicates the number of individuals located in the area $(x, x + dx)$ $(y, y + dy)$.

If the influence of the locality is *not selective* for individuals of certain characteristics F and w, we shall have:

$$N(F, w, x, y) = N_0(F, w)\, N^*(x, y). \tag{22}$$

But, in general, this need not be the case. We may now consider as the fundamental equation for the structure of society:

$$\begin{aligned} a(F, w, x, y) = \alpha w(x, y) + \beta \int (F' - F)\, f(x' - x, y' - y) \\ \times N(F', w', x', y')\, dF'\, dw'\, dx'\, dy', \end{aligned} \tag{23}$$

or similar but more general equations.

The combination of the study of the "geographic" factor with the study of formation of "classes" leads to some very interesting results and formulae. We shall briefly indicate here one case *only as an illustration.*

Consider a case where the whole group of $\mathfrak{N}$ individuals divides naturally, according to equation (18), into only *two* classes. If for such a division we have the inequality

$$\int_0^{F_x} N(F)\, FdF < \int_{F_x}^{F_m} N(F)\, FdF, \tag{24}$$

then the total influence of the class (F_x, F_m), which we shall call "first class," is larger than the total influence of the class $(0, F_x)$, which we shall call "second class." We shall mark all quantities referring to individuals of the first class with a prime, and all quantities referring to individuals of the second class with a double prime. The expression

$$\begin{aligned} \int_{F'=F_x}^{F'=F_m} \int_{F''=0}^{F''=F_x} \int_S \int_S (F' - F'')\, N'(F', x', y')\, N''(F'', x'', y'') \\ \times f(x' - x'', y' - y'')\, dF'\, dF''\, dx'\, dx''\, dy'\, dy'', \end{aligned} \tag{25}$$

in which the integration with respect to x', y', x'' and y'' is taken over the whole region occupied by the total group, can be considered as a measure of the extent to which the second class is influenced or "controlled" by the first. If $N''(F', x'', y'')$ is given, that is, if the distribution of the population of the second class is known, we may ask what $N'(F', x', y')$ should be in order to make expression (25) a maximum. In other words, what should be the distribution of the population of the first class in order to achieve the best control of the second?

A very elementary example of such a problem may be given by considering the special case where two individuals interact only if their distance is below a critical value r_0, and where the distribution of individuals of the second class in space is uniform, with the density q individuals per unit area. Then any group of individuals of the first class, if it is located in a very small region, can control only $\pi r_0^2 q$ in-

dividuals of the second class. But when the first class subdivides into two groups, separated by a distance $r > r_0$, then the number of controlled individuals doubles. The maximum is reached for a subdivision of the first class into

$$n = \frac{S}{\pi r_0^2} \tag{26}$$

separate groups, where S is the total area occupied by the population.

Inasmuch as $f(x' - x, y' - y)$ depends on the technical and engineering facilities at the command of the individuals, the considerations of this section indicate a way of taking those into account. It may happen that at first a "centralized" first class will correspond to an optimum. If S grows later on, it may require a splitting into a number of groups for better control. Later on, with increased technical means (increasing r_0), the number of groups n may again decrease, a "centralization" taking place.

IV

We have considered the influence of the past history of the individual as contained implicitly in w. We shall now consider the possible influence of past history of the whole society on its present state.

For the sake of definiteness and brevity, we shall consider here only one particular case. Let the structure of the group be determined at first by one of the equations of section II, and let it again be divided in two classes, on the basis of equation (18). Let us now consider the development of the society for several generations, under the assumption that the progeny of the first class associates only with the progeny of the same class. That is, instead of the association by actual similarity, we shall have an association by the *similarity of the past generations*. For simplicity let us consider every individual as characterized by only one variable F.

Let us consider separately the distribution function for each class: $N_1(F)$ and $N_2(F)$. Originally we have $N_1(F) + N_2(F) = N(F)$. But, as we shall presently see, due to changes of those functions with time, this relation will not hold in, general, for later times.

The progeny of an individual F^* will not all possess the same characteristic F^*. The values of F will be distributed according to some distribution function $p(F^*, F)$ with a maximum for $F = F^*$. The function $p(F^*, F)$ is the fraction of individuals born of a parent F^* and possessing the characteristic F. By this definition $p(F^*, F)$ is not the same as $p(F, F^*)$. The shape of $p(F^*, F)$ can, in principle, be determined from genetic considerations. We shall leave it unde-

termined at present. Let the birth rate per individual F, be $n(F)$, so that $n(F)\ N_2(F)\, dF$ individuals are born per unit time from parents F of the second class. We then have

$$\int_0^{F_s} p(F, F^*)\, dF^* = n(F)\ N_2(F, t)\ dF. \tag{27}$$

We now write $N_2(F,\ t)$ to indicate that, due to the scattering represented by p, the shape of N changes with time.

The total number of individuals born per unit time from parents F and themselves having $F = F^*$ is equal to

$$n(F)\ N_2(F, t)\ p(F, F^*)\ dF, \tag{28}$$

and the total number of individuals with characteristic F, born per unit of time of any parents of the second class, is

$$\int_0^{F_s} n(F^*)\ N_2(F^*, t)\ p(F, F^*)\ dF^*. \tag{29}$$

If $m(F)$ is the total number of deaths of individuals F per unit time, then the total change of $N(F,\ t)$ is given by

$$\delta N_2(F, t) = \int_0^{F_m} n(F^*)\ N_2(F^*, t)\ p(F, F^*)\ dF^* - m(F) \tag{30}$$

which is a functional equation[1] determining $N(F,\ t)$ for any t.

According to our hypothesis, the progeny of all individuals of the second class continues to belong to the second class, the progeny of all those of the first—to the first. Per unit time, the increase in the number of individuals of the second class who have $F > F_r$, is given by

$$\int_{F_r}^{F_m} \delta N_2(F, t)\ dF = \int_{F_r}^{F_m} dF \int_0^{F_s} n(F^*)\ N_2(F^*, t)\ p(F, F^*)\ dF^*$$
$$- \int_{F_r}^{F_m} m(F)\ dF = \phi_2(F_r\,, t). \tag{31}$$

A similar expression is found for the increase $\phi_1(F_r\,,\ t)$ in the number of individuals of the first class with $F < F_r$. We have taken inequality (24) as the condition of the possibility for the first class to control the second. Due to the change represented by equation (31), the inequality (24) will finally cease to hold and the social structure will become unstable, the influence of the first class having been weakened, that of the second class increased. A rearrangement is bound to occur. At what time t this instability will occur is deter-

mined by the equation (31) together with a similar one for the first class, combined with inequality (24).

We may consider the stability of a social structure from a different point of view. Whereas originally we had $a = a(F)$, we now find that individuals with the same F may have different a's because they belong to different classes. To a given value of F now corresponds a multitude of values of a, which will be determined by a distribution function

$$u(F, a)\, da\,, \tag{32}$$

giving the number of individuals with the characteristic F and having an a between a and $a + da$. We have:

$$\int_0^{a_m} u\, da = N(F, t). \tag{33}$$

The function u itself is determined by equation (31) and is a function of t, also. We find that now the distribution of a no longer corresponds to the "natural" distribution, as given by one of the equations of section II, where a was determined by F only. Let $a(F)$ be that "natural" distribution. Then the *total* deviation of the social structure at the moment t from the "natural" one, prevailing at $t = 0$, is given by:

$$\int_0^{F_m} dF \int_0^{a_m} u(F, a)\,[a(F) - a]^2\, da = \psi(t). \tag{34}$$

We may consider as a criterion of stability that this total deviation should not exceed a given value h; in other words, $\psi(t) \leqq h$. The moment t, at which instability occurs, is given by the root of the equation $\psi(t) = h$.

A still different criterion for the stability of a social structure may be considered. In the absence of other individuals, the activity a of an individual is determined only by w, being given by $a(F) = \alpha w(F)$. Due to the interaction with other individuals, $a(F)$ does differ from $\alpha w(F)$. We may postulate that the sum total for the community of the differences $[a(F) - \alpha w(F)]^2$ should not exceed a threshold h. Since $a(F)$ is given by one of our fundamental equations, we have another integral condition for stability, which depends on $N(F, t)$ and therefore changes with time.

In ordinary language, we may state this criterion by saying that society becomes unstable when the differences between the natural desires of the individuals and the actual fulfillment of those desires become excessive.[2]

We may also consider an intermediate case, where any individual with $F > F_x$, born in the second class, can eventually pass into the first, but where a resistance is offered to such a passage. This resistance will be, in general, a function of $F - F_x$. The time τ_{21} which it takes for an individual to pass from the second class into the first is also a function of $F - F_x$. We express this by writing $\tau_{21}(F - F_x)$. Its inverse can be considered as a measure of the corresponding "social mobility."[3] The total number of individuals passing from the second class into the first per unit time is given by

$$\int_{F_x}^{F_m} \frac{N_2(F, t)}{\tau(F - F_x)} dF = M_{21}(F_x, t) . \tag{35}$$

A similar expression for M_{12} will give the total number of individuals born in the first class with $F < F_x$, passing per unit time into the second class. If the instability discussed above should never occur in the course of history of a society, we must have:

$$\begin{aligned} \phi_{21}(F_x, t) &\leqq M_{21}(F_x, t) ; \\ \phi_{12}(F_x, t) &\leqq M_{12}(F_x, t) ; \end{aligned} \tag{36}$$

ϕ being defined by equation (31), for then the "social mobility" will always level out the deviations from the "natural distribution."

Besides the type of instability discussed above, a different kind may also happen. We have seen in section II that the structure of the social group may be characterized by an integral equation (equations 8 and 9). If, as will be the simplest case, those equations are linear, as equations (8) and (9), they will possess only one solution. We must in general, however, consider the possibility of non-linear equations also. In that case it is possible that there will be two or more solutions. The situation is not without formal analogy to physicochemical systems with several equilibria. Sudden transitions from one state into the other one will have to be considered.

V

The question now arises as to how to describe mathematically those transitory states caused by one of the above discussed factors. In other words, when for instance a state in which $\psi(t) > h$ (equation 34) is reached, how will the transition towards the "natural" state for which $\psi = 0$ take place?

We must introduce more general equations covering the "dynamic" cases, also. They must be such that the above-discussed equations

of section II would be particular cases of the more general ones for "stationary states." A simple case would be, for instance, to put, with A as a constant,

$$\frac{da(F,I,w,t)}{dt} = -A\ [a(F,I,w,t) - aw$$

$$-\beta \int_0^\infty N(F',I',w')\,|F'-F|\,w'\,dF'\,dI'\,dw' \qquad (37)$$

$$-\gamma I \int_0^\infty N(F',I',w')F'a'(F',I',w')\,dF'\,dI'\,dw'].$$

This leads to equation (9) for a stationary state, for which $da/dt = 0$.

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CHAPTER II

GENERAL MATHEMATICAL CONSIDERATIONS CONTINUED

In chapter i we have outlined a general mathematical approach to a deductive, theoretical science of human society. We have established various general forms of equations which can be used for that purpose, and indicated the way of describing in mathematical terms some phenomena of social interactions. In those fundamental equations we have been making use of some quantities which are not directly accessible to physical measurements. We justified this procedure, however, by invoking the example of physical sciences, where equations are established between quantities which may be directly inaccessible to measurements, but which are measurable "in principle." This "measurability in principle" is made possible by the application of the mathematical equations which eventually give us connections between those directly unmeasurable quantities and others which can be measured directly. We shall now discuss the way by which, in the ultimate development of the theoretical system, a measurement and determination "in principle" of the quantities involved may be achieved.

We must, however, emphasize that such a comparison in principle does not necessarily mean a comparison with *actual existing data.* In the early stages of a deductive, theoretical science, we must study *at first* purely imaginary cases which, due to the intentional oversimplification, have no real existence. If we speak here of a comparison of an equation with *observable* data, we thereby mean this: we consider a simple, *theoretically possible*, but not an actual, case, and set up equations which describe it. The equations themselves may be of such a nature that even if the case studied actually existed, they would not be directly verifiable. But some of their consequences could be compared with observable data, *which would be available if our case really existed.*

I

We shall begin by considering the simplest case of one activity only, represented by equation (5) of chapter i, and first simplify the

case still further by considering that all individuals have the same desire w for the activity i, and differ only in their coefficients of influence F. The activity i itself may be of any kind: production of food or other practical necessities, or any artistic activity, such as painting, composing, etc. The proper practical unit for the intensity a_i may vary from case to case. But, in principle, for a given activity its intensity a_i may be expressed by the amount of energy spent on it on the average per unit of time. Since in the absence of the influence of others we have:

$$a_i = \alpha w \, , \tag{1}$$

where α is the coefficient of proportionality, dependent on the choice of units, we may measure w by the amount of a_i which an individual does of his own free will. Then the dimensions of both a_i and w are

$$[m]\,[l]^2\,[t]^{-3} \tag{2}$$

and, since β in equation (5) of chapter i is also only a coefficient of proportionality, F becomes a pure number. Putting $\alpha = \beta = 1$ in equation (5) of chapter i, and remembering that in the present case $w = w'$, we find for two individuals

$$\begin{aligned} a_i &= w[1 + (F' - F)] \, , \text{ if } F' > F; \\ a_i &= w \quad \text{if } F' \leqq F. \end{aligned} \tag{3}$$

We see that the difference of the coefficients of influence of two individuals equals 1 if one of them can, through his influence on another, induce him to double the amount of the latter's activity done of free will.

Of course, we must remember that this holds only for the particular hypothetical case which we decided to consider here. Adoption of different postulates will require a modification of the setup of units, but the procedure remains fundamentally the same.

We shall now illustrate on an arbitrarily chosen function $N(F, w)$, which now degenerates into $N(F)$, how we can obtain equations which determine the not directly measurable quantities. Again the proper procedure would be to investigate various possible forms of $N(F)$, and to begin with the simplest cases. One may think that a normal distribution should be investigated first. In view of its wide range of applications in statistics, a normal distribution certainly deserves a separate study. In view of the fact, however, that the evaluation of some integrals in finite form becomes impossible for a normal distribution, we shall consider here a different one, namely:

$$N(F) = AF\, e^{-aF}, \tag{4}$$

where A and a are positive constants. This function $N(F)$ is equal to zero for $F=0$ and $F=\infty$, has a maximum for $F=1/a$, and an inflection point for $F=2/a$.

From the requirement that

$$\int_0^\infty N(F)\,dF = \mathfrak{N}\,, \tag{5}$$

where $\mathfrak{N}$ is the total number of individuals, we find

$$A = \mathfrak{N}a^2, \tag{6}$$

so that

$$N(F) = \mathfrak{N}a^2 F e^{-aF}\,. \tag{7}$$

It may be argued that the integration in equation (5) should be carried out not from zero to infinity, but from zero to F_m, F_m being the maximum value of F that occurs in the population. This would give

$$A = \mathfrak{N}/[1/a^2 - (F_m + 1/a)\,e^{-aF_m}/a]\,, \tag{8}$$

which reduces to equation (6) if F_m is so large that e^{-aF_m} is very small. We shall confine ourselves to such a case and use expression (6) instead of (8).

If we introduce expression (7) into equation (18) of chapter i which, written explicitly, reads:

$$\int_{F_r}^{F_m}\int_{F_r}^{F_m} [(F'-F)^2 - \Delta^2]\,N(F)N(F')\,dF\,dF' = 0\,, \tag{9}$$

we can perform the necessary integrations and thus obtain a relation between F_r, F_m, Δ, and the parameter a of the distribution function. The final result of the integrations is a rather cumbersome transcendental equation and shall not be given here explicitly. The important thing is that we do obtain a relation of the form

$$U(F_r, F_m, \Delta, a) = 0\,. \tag{10}$$

For the same reason for which we use expression (6) instead of (8), we may use ∞ for the upper limits of integrations in equation (9). In that case equation (10) degenerates into one not containing F_m:

$$U_1(F_r, \Delta, a) = 0\,. \tag{11}$$

Another relation involving F_r is obtained by considering the ratio ϑ of the number of individuals of the upper class to the total number $\mathfrak{N}$ of individuals in the society. Again using infinity instead of F_m for the upper limit of integration, we have

$$\vartheta = \frac{1}{\mathfrak{N}} \int_{F_x}^{\infty} N(F)\, dF. \tag{12}$$

Introducing expression (7) into (12) we find:

$$\vartheta = (aF_x + 1)\, e^{-aF_x}. \tag{13}$$

Equation (13) gives a relation between F_x, a and ϑ. The latter is directly measurable. For the case $aF_x > 1$, equation (13) may be simplified to:

$$\vartheta = aF_x\, e^{-aF_x}, \tag{14}$$

or

$$\log \vartheta = \log aF_x - aF_x. \tag{15}$$

Neglecting $\log aF_x$ as compared with aF_x, we find approximately:

$$F_x = -\frac{1}{a} \log \vartheta. \tag{16}$$

A relation between F_x, a and w, which we consider here as constant, is provided by expressing the total intensity A of activity of the whole group. Equation (3) expresses the intensity of a_i of a single individual as a function $a_i(F, F')$ of F and F'. Under the influence of *all* the other individuals, the individual with a given F exhibits an intensity $\boldsymbol{a}(F)$ of activity, equal to

$$\boldsymbol{a}(F) = \int_{F}^{\infty} a_i(F, F') N(F')\, dF'. \tag{17}$$

The total intensity A is given by

$$A = \int_0^{\infty} \boldsymbol{a}(F) N(F)\, dF = \int_0^{\infty} N(F)\, dF \int_F^{\infty} a_i(F, F') N(F')\, dF'. \tag{18}$$

Introducing expression (3) into (18) we find

$$A = \int_0^{\infty} N(F)\, dF \left[w \int_F^{\infty} N(F')\, dF' + \int_F^{\infty} (F' - F) N(F')\, dF' \right]. \tag{19}$$

With $N(F)$ given by expression (7), the right side of (19) may be easily evaluated.

Finally we may compute the intensity A_1 of activity of the upper class only. This is given by

$$A_1 = \int_{F_x}^{\infty} N(F)\, dF \left[w \int_F^{\infty} N(F')\, dF' + \int_F^{\infty} (F' - F) N(F')\, dF' \right]. \tag{20}$$

Expressions (10), (16), (19) and (20) provide us with four

equations connecting the quantities F_x, F_m, a, Δ, ϑ, $\mathfrak{R}$, A, A_1 and w. The intensity w of desire for an activity is directly measurable. In fact, with the choice of units which leads to equation (3), w is numerically equal to a, in the absence of any influence by other individuals. The quantities ϑ, $\mathfrak{R}$, A and A_1 are also directly measurable. By means of the four equations, (10), (16), (19) and (20), we may express the not directly measurable quantities F_a, F_m, a and Δ in terms of the other, and thus determine those directly unmeasurable quantities through our equations.

Instead of expression (7), we could have chosen any other *one-parametric* distribution function. If we choose a function with more than one parameter, we would have to establish additional equations for the determination of those parameters.

In principle it may be possible to find more equations than there are unknowns. This would provide for a method of checking the correctness of the hypothetical choice of the function $N(F)$ and of other assumptions. If the choice of $N(F)$ is correct, and all the assumptions correct, then the values of the same unknown quantity, computed from different equations, should be the same.

II

Now let us investigate somewhat closer the function $N(F)$ itself. In chapter i we have considered the variation of $N(F)$ due to the fact that the progeny of an individual with a definite F may have in general a different value of F. Again let $p(F^*, F)$ be the number of individuals having a characteristic F and born of parents F^*. In general we must consider the case in which two parents have a different F, but for the time being we confine ourselves to the simpler case in which both parents have an identical F. The total number of individuals with the characteristic F, born of any parents per unit time, is [chapter i, equation (29)]

$$\int_0^\infty n(F^*)\, N(F^*, t)\, p(F^*, F)\, dF^*, \tag{21}$$

where $n(F^*)$ denotes the birth rate per individual. If $m(F)$ is the death rate, also per individual, we have for the total change of $N(F, t)$ per unit time

$$\frac{\partial N(F, t)}{\partial t} = \int_0^\infty n(F^*) N(F^*, t)\, p(F^*, F)\, dF^* - m(F) N(F). \tag{22}$$

Let us consider the simplest case, that both $n(F)$ and $m(F)$ are con-

stants, that is that the birth and death rates are the same for all types of individuals. Then (22) becomes:

$$\frac{\partial N(F,t)}{\partial t} = n \int_0^\infty N(F^*,t)\, p(F^*,F)\, dF^* - mN(F). \qquad (23)$$

We shall solve equation (23) by putting

$$N(F,t) = N^*(F)\, \phi(t), \qquad (24)$$

where $N^*(F)$ is a function of F only, and $\phi(t)$ is a function of t only. We shall determine $N_0(F)$ and $\phi(t)$ so as to satisfy both equations (23) and the requirement that at an initial moment t, which we may put without any loss of generality equal to zero, $N(F,t)$ should be a given function $N_0(F)$ of F.

Introducing equation (24) into (23) we find:

$$N^*(F)\frac{d\phi}{dt} = n\phi(t) \int_0^\infty N^*(F^*)\, p(F^*,F)\, dF^* - mN^*(F)\phi(t). \qquad (25)$$

Putting

$$\frac{n\int_0^\infty N^*(F)p(F^*,F)dF^*}{N^*(F)} - m = \mu, \qquad (26)$$

equation (25) becomes

$$\frac{d\phi}{dt} = \mu\phi, \qquad (27)$$

which gives

$$\phi = A_0 e^{\mu t}, \qquad (28)$$

A_0 being a constant of integration.

For every given value of μ, equation (26) gives us an equation for the determination of $N^*(F)$. Expression (26) can be written after simple rearrangements in the following way:

$$N^*(F) = \frac{n}{\mu + m}\int_0^\infty N^*(F^*)p(F^*,F)dF, \qquad (29)$$

which is a homogeneous integral equation of second kind, with the kernel $p(F^*,F)$. Equation (29) in general possesses solutions only for definite values of the constant

$$\lambda = \frac{n}{\mu + m}. \qquad (30)$$

Let $\lambda_1, \lambda_2, \cdots, \lambda_i$ be those "eigenvalues" arranged in increasing order so that:

$$\lambda_1 < \lambda_2 < \lambda_3 < \cdots. \tag{31}$$

Then in order that equation (29) should have solutions at all, μ must have one of the values

$$\mu_i = \frac{n - \lambda_i m}{\lambda_i} = \frac{n}{\lambda_i} - m. \tag{32}$$

If $N_i^*(F)$ is an "eigenfunction" of equation (29), corresponding to the "eigenvalue" λ_i, then:

$$A_i N_i^*(F)\, e^{\mu_i t}$$

is a particular solution of equation (23), and the general solution is given by:

$$N(F, t) = \sum_1^\infty A_i N_i^*(F)\, e^{\mu_i t}. \tag{33}$$

For $t = 0$, this is equal to

$$N(F, 0) = \sum_1^\infty A_i N_i^*(F), \tag{34}$$

and since all $N_i^*(F)$ form a complete orthogonal system, the coefficients A_i can be determined so that

$$\sum_1^\infty A_i N_i^*(F) = N_0(F). \tag{35}$$

We have

$$A_i = \int_0^\infty N_0(F) N_i^*(F)\, dF. \tag{36}$$

Equations (33) and (36) represent the general solution of equation (23). Let us consider some of the consequences.

Because of equation (33) we have:

$$\mu_1 > \mu_2 > \mu_3 > \cdots; \quad \mu_\infty = -m. \tag{37}$$

If

$$m > \frac{n}{\lambda_i}, \tag{38}$$

then all μ's are negative. In that case, regardless of the choice of the coefficients A_i, in other words, regardless of the initial distribution $N_0(F)$, the expression $N(F, t)$ given by equation (33) tends to zero. Expression (38) therefore sets an upper limit for the death rate m, above which the social group will, with time, become extinct.

If, however,

$$m < \frac{n}{\lambda_1},$$

then some μ's, say $\mu_1, \mu_2 \cdots \mu_s$, will be positive, others, $\mu_{s+1}, \mu_{s+2}, \cdots$, will be negative. But then all terms in equation (33) above the sth will tend to zero, while the first s terms will increase. But with increasing t the term with the largest μ, that is the first term, will exceed the others more and more, the ratio

$$e^{\mu_i t}/e^{\mu_1 t} \qquad (i \leqq s)$$

tending to zero. Hence after a sufficient time has elapsed, $N(F,t)$ will be given by

$$N(F,t) = A_1 N_1^*(F)\, e^{\mu_1 t}. \tag{39}$$

Equation (39) shows that the total number $\mathfrak{N}$ of individuals will increase exponentially, but the distribution function $N(F)$ will not vary, being given by the first "eigenfunction" $N_i^*(F)$ of the integral equation (29). A similar result holds for the special case where $m = \frac{n}{\lambda_1}$. Then $\mu_1 = 0$, all others are negative, and expression (33) tends asymptotically to $A_1 N_1^*(F)$. In this case not only does the distribution function tend asymptotically to a fixed form, but the total number of individuals also tends to be a constant.

We thus find a fundamental result: under the simplified assumptions made here, the distribution function $N(F)$ always tends either to zero or to a stationary distribution which is determined by the function $p(F^*, F)$, since the latter is the kernel of the integral equation (29) whose first "eigenfunction" determines the stationary distribution. Any disturbances such as wars, starvation, etc., may upset this distribution temporarily, but in time it will again be restored.

If equation (29) possesses solutions within a continuous range of values of λ, or for any value of λ, the case must be treated differently.

In chapter i, we learned how the variation of $N(F)$ with time determines the variation of the social structure and eventually causes its instability. Observations of the variations in time of the social structure may also lead us to equations which connect some of the directly unobservable quantities with directly measurable ones. For instance, as we have seen in chapter i, due to variation of $N(F)$, the second class will contain a certain number of individuals with $F > F_x$. If the variation of $N(F)$ with respect to time is given, then this number R of individuals with $F > F_x$ is also given for any moment, $R = R(t)$. These individuals will not be controlled in a normal way

by the first class. If, as is usually the case, the controlling class uses various methods of coercion against active political opponents, then $R(t)$ would represent the number of individuals subject to such coercion. This number can be directly determined. If this known number is N_c, then

$$R(t) = N_c \tag{40}$$

gives us an equation, involving some parameters, which determines the variation of $N(F)$, $p(F^*, F)$, etc. Together with other possible equations, it may be used to calculate these parameters.

III

In the general case the coefficient of influence may itself be a derivative notion. If an individual can easily perform and does perform an activity the results of which are badly needed by other individuals, then the first individual may have a strong control over the second. We may thus consider the coefficients of influence as being functions of the activities. For the case of two activities the situation is mathematically represented in the following manner.

Let each individual be characterized by the desires w_1 and w_2 for the performance of the activities a_1 and a_2, and by the desires u and u_2 to possess the results of the corresponding activities without actually performing them.

Then

$$\begin{aligned} a_1(w_1, u_1, w_2, u_2) &= \alpha w_1 \\ &+ \beta u_2 \int_0^\infty N(w'_1, u'_1, w'_2, u'_2) a_2(w'_1, u'_1, w'_2, u'_2) \\ &\qquad u'_1 \, dw'_1 \, du'_1 \, dw'_2 \, du'_2 ; \\ a_2(w_1, u_1, w_2, u_2) &= \alpha w_2 \\ &+ \beta u_1 \int_0^\infty N(w'_1, u'_1, w'_2, u'_2) a_1(w'_1, u'_1, w'_2, u'_2) \\ &\qquad u'_2 \, dw'_1 \, du'_1 \, dw'_2 \, du'_2 . \end{aligned} \tag{41}$$

The integrals in the right-hand side of both equations are constants which we denote by A_1 and A_2 respectively. Hence

$$\begin{aligned} a_1 &= \alpha w_1 + A_1 \beta u_2 , \\ a_2 &= \alpha w_2 + A_2 \beta u_1 . \end{aligned} \tag{42}$$

The constants A_1 and A_2 are determined from the two equations

$$
\begin{aligned}
A_1 &= \int_0^\infty N(w_1, u_1, w_2, u_2)\, u_1 (\alpha w_2 + A_2 \beta u_1)\, dw_1\, du_1\, dw_2\, du_2 , \\
A_2 &= \int_0^\infty N(w_1, u_1, w_2, u_2)\, u_2 (\alpha w_1 + A_2 \beta u_2)\, dw_1\, du_1\, dw_2\, du_2 .
\end{aligned} \tag{43}
$$

CHAPTER III

AN APPROXIMATE TREATMENT OF THE INTERACTION OF SOCIAL CLASSES

In the preceding chapters we developed a general mathematical approach to social phenomena. It has been shown how some general characteristics of sociological changes leading to instabilities can be described mathematically. However, the equations involved are of a type that do not lend themselves readily to closed solutions, even in simplest cases. As we attempt to describe more complex situations, the mathematical difficulties increase. The use of some general approximation method is therefore indicated.[1]

We have seen that for a given distribution function of some individual characteristic in a given population, this population will split into two or more classes due to a tendency of any individual to associate preponderantly with other individuals possessing a similar characteristic. In the special case where this characteristic is the "coefficient of influence" and where the number of classes is two, we have one class influencing and controlling the activities of the other. The exact interaction of the two classes, which are not homogeneous with regard to the given characteristic of the individuals constituting the class, depends on the structure of the classes. This structure is itself determined by the distribution function of the characteristic considered. We obtain a great simplification if we introduce, instead of the actual structure of the classes, an average value of the characteristic of the individuals of a given class. We thus have a class of individuals having a certain characteristic in the amount a_1, another having the same characteristic in the amount a_2, etc. Or, more generally, we may have several classes, characterized by different average amounts of a characteristic a, several other classes characterized by different average amounts of *another* characteristic b, etc. For instance a may represent an average amount of education, while b may represent an average ability to play football. Such classes may or may not overlap. That is, an individual may belong to the class of highly educated people and of average quality football players.

We shall here confine ourselves to the special case where the characteristic of the individual is his ability to influence activities of other individuals. As we have seen, this ability is itself a composite one.

The influence exerted may be due to physical force, to moral persuasion, wealth, etc. We shall consider here at first only a *purely theoretical case* which presents some interest. Similar considerations may then be applied later on to other different cases.

In actual life, we find that a rather large number of individuals choose their activities not because of any firm conviction of the necessity of such activity, nor because of a particularly strong inclination for this activity, but merely because the majority of the other individuals with whom they come into contact perform this activity. There are, of course, all gradations from individuals who would never do anything that is not done "by everybody" to those who are guided only by their own convictions and do not care in the least what "people may say about them." Let us first consider, for the sake of simplicity, an abstract, non-existing situation in which we have two *distinct* groups of individuals, one composed of individuals who act *only* on their own initiative (type I) and the other composed of individuals who are strongly influenced by their fellow-individuals and direct their activities exclusively according to what others do (type II). As remarked above, this is a fictitious case, but it may turn out to be a first approximation to a more real case when there are gradations in the above characteristics.

For convenience, we shall refer to individuals of type I as the "active" individuals, and to those of the type II as the "passive" ones. The meanings of the two words are here different from the customary. By definition, Leo Tolstoy and Mr. M. K. Gandhi are both "active" individuals. Tolstoy's "non-resistance to evil" or Gandhi's "passive disobedience" are here considered as a type of activity which may be imposed upon others.

An individual may be active in one respect and passive in another. Thus while an individual may have very definite musical tastes or political views which may differ from those of his neighbors, he may at the same time wear a conventional hat, or a necktie, just because "everybody else does it."

Let us consider a total number N of individuals forming a given population. Of these N individuals, let a number x_0 belong to the active type, being characterized by a certain activity A. We shall refer to them as individuals I_A. Let y_0 individuals belong also to the active type, but let the activity of their choice be a different one, B, such that A excludes B. We shall refer to them as individuals I_B. We may ask what will be the behavior of the remaining

$$N' = N - x_0 - y_0 \tag{1}$$

individuals whom we assume to be all of the passive type. The an-

swer to that question will, of course, depend on the assumptions which we make concerning the mechanism of influence of some individuals upon others. Let us again, for the sake of simplicity, make one of the simplest possible assumptions, namely, that an individual of the passive type is more strongly inclined to behave in a given way the more frequently he comes into contact with active individuals who exhibit this particular behavior. Let each individual of type I_A actively try to influence as many passive individuals as possible by actually coming into contact with as many of them as he can. Then, denoting by x the number of passive individuals who exhibit the behavior A because they were influenced to do so by the individuals of the type I_A, we see that x *will increase at a rate* which is approximately proportional to x_0. Let the coefficient of proportionality be a_0 so that the total contribution of the individuals of type I_A to the increase of x is a_0x_0. But the x individuals who now exhibit the behavior A, also come into contact with other passive individuals who have not yet adopted the behavior A. Therefore, they will also contribute to the rate of increase of x, in the amount ax, a being in general different from a_0. Similarly, denoting by y the number of passive individuals who have adopted the behavior B through contact with other individuals, we find that they contribute to the rate of increase of y an amount ay while the individuals I_B contribute the amount c_0y_0, c_0 being a coefficient of proportionality. The coefficient a is assumed to be the same as before, since in both cases the coefficient refers to passive individuals.

If there is only one alternative for behavior, either A or B, then the increase of x is accompanied by a corresponding decrease of y, and *vice versa*. Therefore, we have for the total rate of increase of x:

$$\frac{dx}{dt} = a_0x_0 + ax - c_0y_0 - ay; \tag{2}$$

and, correspondingly:

$$\frac{dy}{dt} = c_0y_0 + ay - a_0x_0 - ax. \tag{3}$$

Since, by definition of x and y, and because of (1),

$$x + y = N - x_0 - y_0 = N', \tag{4}$$

the system (2) and (3) reduces actually to one equation, either for x or for y. Let us consider the equation in x.

From expression (4), we have

$$x = N' - y; \quad y = N' - x. \tag{5}$$

Introducing this into equation (2), we find

$$\frac{dx}{dt} = 2ax - (aN' - a_0 x_0 + c_0 y_0) . \tag{6}$$

If

$$aN' - a_0 x_0 + c_0 y_0 < 0 , \tag{7}$$

then for $x = 0$, the derivative $dx/dt > 0$ and it remains positive as x increases. In other words, if inequality (7) is satisfied, then if originally all individuals except x_0 showed the behavior B, the number of individuals showing behavior A will increase until all except y_0 will exhibit behavior A. Introducing for N' the expression (1), we find that inequality (7) is equivalent to

$$x_0 > \frac{a}{a_0 + a} N + \frac{c_0 - a}{a_0 + a} y_0 . \tag{8}$$

By a similar argument, starting with equation (3), we find that if

$$y_0 > \frac{a}{c_0 + a} N + \frac{a_0 - a}{c_0 + a} x_0 \tag{9}$$

then for $y = 0$, the derivative $dy/dt > 0$ and remains positive as y increases. When inequality (9) holds, then even if originally all individuals except y_0 exhibited behavior A, they will all, except the number x_0, eventually exhibit behavior B. If inequality (8) is not satisfied, it does not mean that (9) is satisfied, and *vice versa*. Solving (8) with respect to y_0 gives

$$y_0 < \frac{a_0 + a}{c_0 - a} x_0 - \frac{a}{c_0 - a} N ; \tag{10}$$

while solving inequality (9) with respect to x_0 gives

$$x_0 < \frac{c_0 + a}{a_0 - a} y_0 - \frac{a}{a_0 - a} N . \tag{11}$$

Thus, it may happen that neither inequality (8) nor (9) is satisfied. When this is the case, then for $x = 0$, $dx/dt < 0$; but since x cannot be negative, this simply means that if all individuals except x_0 exhibit behavior B, this social configuration is relatively stable and nothing happens to change it. Similarly, if all individuals except y_0 exhibit behavior A, this is also a stable configuration. The social aggregate thus possesses two stable configurations of equilibrium: one (configuration A) in which all except y_0 exhibit behavior A, another (configuration B) in which all except x_0 exhibit behavior B. In order

to bring the social aggregate from configuration B into configuration A, some external disturbance (e.g. intervention of a different social aggregate, war) must make

$$x > \frac{aN' - a_0 x_0 + c_0 y_0}{2a} = \frac{aN - (a_0 + a)x_0 + (c_0 - a)y_0}{2a}. \tag{12}$$

Then, as is readily seen from equation (6), dx/dt becomes greater than zero, and x will increase until it becomes equal to $N' = N - x_0 - y_0$. Similarly, in order to bring the social aggregate from configuration A into configuration B, an external disturbance must make

$$y > \frac{aN' - c_0 y_0 + a_0 x_0}{2a} = \frac{aN - (c_0 + a)y_0 + (a_0 - a)x_0}{2a}. \tag{13}$$

While neither inequality (8) nor (9) may be satisfied, it cannot happen that they both are satisfied. Thus when (8) is satisfied, the only stable configuration is A; when (9) is satisfied, the only stable configuration is B; when neither (8) nor (9) is satisfied, there are two stable configurations, A and B.

Let us analyze the meaning of the coefficients a_0, c_0, a and of the inequalities (8) and (9). The coefficient a_0 is the average number of passive individuals with whom every individual of type I_A comes into contact per unit time. The quantity c_0 has a similar meaning with respect to individuals I_B, while a is the average number of passive individuals with whom every passive individual comes in contact per unit time. All three are definitely measurable quantities. The individuals may come in contact with each other directly, or indirectly through mail, press or radio. Thus a_0, c_0, and a are influenced by any developments of means of communications, in the broader meaning of this word. These coefficients may be expressed as functions of the average circulation of mail m, average circulation of newspapers n, number of broadcasting stations b, and number of radio receivers r. Thus

$$\begin{aligned} a_0 &= f_1(m, n, b, r); \\ c_0 &= f_2(m, n, b, r); \\ a &= f_3(m, n, b, r). \end{aligned} \tag{14}$$

If the groups I_A and I_B are particularly eager to induce the passive individuals to exhibit their respective behaviors, they would endeavor to increase a_0 and c_0 by varying appropriately m, n, b and r as far as they can.

This suggests the investigation of the case in which

$$a_0 \sim c_0; \quad a_0 >> a; \quad a_0 >> a. \tag{15}$$

In this case, the expressions

$$\frac{a}{a_0 + a} \quad \text{and} \quad \frac{a}{c_0 + a} \tag{16}$$

in inequalities (8) and (9) are much smaller than unity, while

$$\frac{c_0 - a}{a_0 + a} \quad \text{and} \quad \frac{a_0 - a}{c_0 + a} \tag{17}$$

are of the order of magnitude of unity. Both x_0 and y_0 may be rather small fractions of N. If, for instance, inequality (8) is satisfied, so that the society is in configuration A, and then x_0 gradually decreases or y_0 increases, so that finally inequality (9) becomes satisfied, then at that moment y will begin to increase until the whole society passes into configuration B. If both x_0 and y_0 are small fractions of N, then a relatively small number of individuals added to either x_0 or y_0 may produce a sudden change in the behavior of the whole society. A numerical example will illustrate this. First let $a_0 = c_0 = 1{,}000$ individuals per day, while $a = 10$ individuals per day. Let N be 10^7 individuals and let y_0 be 10^5 individuals so that $y_0 = 0{\cdot}01\, N$. In this case, inequality (8) gives:

$$x_0 > 1{\cdot}97 \times 10^5. \tag{18}$$

Let $x_0 = 2 \times 10^5$ individuals, so that inequality (18) is satisfied, and the whole social aggregate except y_0 individuals exhibits behavior A. Now let x_0 decrease to 10^5, for one reason or another, while y_0 increases to 2×10^5. Then inequality (18) will cease to be satisfied, while (9) becomes satisfied and the configuration begins to change from A to B. Thus a change of 100,000 in a population of 10,000,000 produces a complete change in the behavior of the whole population.

The dynamics of the transition from configuration A to the configuration B, and inversely, is given by equations (2) and (3). Consider, for instance, the transition from B to A. Putting

$$aN' - a_0 x_0 + c_0 y_0 = -u, \tag{19}$$

equation (6) becomes:

$$\frac{dx}{dt} = 2ax + u. \tag{20}$$

This gives

$$x = \frac{C}{2a} e^{2at} - \frac{u}{2a},$$

C being a constant of integration. If for $t = 0$, $x = 0$, then

$$C = u$$

and hence

$$x = \frac{u}{2a}(e^{2at} - 1). \tag{21}$$

The configuration A is reached when $x = N - x_0 - y_0$. Hence the moment t_A at which this occurs is determined by

$$\frac{u}{2a}(e^{2at_A} - 1) = N - x_0 - y_0, \tag{22}$$

or

$$t_A = \frac{1}{2a}\log\frac{2a(N - x_0 - y_0) + u}{u}. \tag{23}$$

With the values for a_0, c_0, a and x_0 used above, we find that t_A is of the order of a day. This agrees in general with the rapid spread of all kinds of mass hysterias, revolts. etc.

As we already remarked, the coefficients a_0, c_0 and a are functions of various parameters characterizing technical facilities at the disposal of the individual. With increasing technical facilities, they also increase, thus shortening t_A. These coefficients are also functions of the personal endeavor of the individuals to influence others. In general, we would expect that the element of personal effort will be more strongly pronounced in active individuals than in the passive ones. Hence, while a will be relatively constant for constant m, n, b, r (cf. equations 14), a_0 and c_0 will vary within a rather wide range. This need not necessarily be so. But for definiteness, let us consider here this particular case without any prejudice against the other possibilities. In this case, a_0 and c_0 might well be called "coefficients of propaganda."

It may be suggested in passing that the above type of interactions may be applied to the changes in fashions or fads which actually do come and go rather suddenly.

Let us consider the case in which the efforts of individuals I_A in influencing passive individuals decrease as the number of successfully influenced individuals increases. This is psychologically a rather plausible situation. With increasing success, when its permanence appears assured, some people are apt to decrease their efforts. In this case a_0 will be a decreasing function of x. In the simplest case it will be a linear function, so that

$$a_0 = a_0^*(1 - \varepsilon x). \tag{24}$$

Similarly, we put

$$c_0 = c_0^*(1 - \varepsilon' y). \tag{25}$$

After rearrangement, equation (6) now becomes, because of expression (5):

$$\frac{dx}{dt} = (2a - a_0^* \varepsilon x_0 - c_0^* \varepsilon' y_0) x$$
$$- [(a - c_0^* \varepsilon' y_0) N' - a_0^* x_0 + c_0^* y_0]. \tag{26}$$

If the expression in brackets is negative, which, because of (1), is equivalent to

$$x_0 > \frac{a - c_0^* \varepsilon' y_0}{a_0^* + a - c_0^* \varepsilon' y_0} N + \frac{c_0^* \varepsilon' y_0 + c_0^* - a}{a_0^* + a - c_0^* \varepsilon' y_0} y_0 = X_1; \tag{27}$$

then for $x = 0$, $dx/dt > 0$, and the influence of the x_0 individuals I_A increases. However, it may happen that while inequality (27) is satisfied, the coefficient of x in equation (26) may be either negative or positive. If it is negative, we have

$$x_0 > \frac{2a - c_0^* \varepsilon' y_0}{a_0^* \varepsilon} = X_2. \tag{28}$$

Putting

$$2a - a_0^* \varepsilon x_0 - c_0^* \varepsilon' y_0 = C_1; \tag{29}$$

$$(a - c_0^* \varepsilon' y_0) N' - a_0^* x_0 + c_0^* y_0 = C_2; \tag{30}$$

equation (26) becomes

$$\frac{dx}{dt} = C_1 x - C_2. \tag{31}$$

If both inequalities (27) and (28) are satisfied, then $C_1 < 0$; $C_2 < 0$. For $x = 0$, $dx/dt > 0$, but $dx/dt = 0$ for

$$x = \frac{C_2}{C_1} = \frac{a_0^* x_0 - c_0^* y_0 - (a - c_0^* \varepsilon' y_0) N'}{a_0^* \varepsilon x_0 + c_0^* \varepsilon' y_0 - 2a} > 0. \tag{32}$$

For $x > C_2/C_1$, $dx/dt < 0$. Hence, in this case, we have a configuration of stable equilibrium, defined by equation (32), in which a part of the passive individuals exhibits behavior A, another part exhibits behavior B. This is different from the previously studied case where, in a stable equilibrium, either all passive individuals exhibited behavior A or all of them exhibited behavior B. The ratio x/y of passive individuals exhibiting behavior A to those exhibiting behavior B, shifts continuously in one direction or another as x_0 and y_0 vary continuously, but, as we shall see presently, only within a certain range of x_0/y_0. Within that range there are no sudden changes as long as

inequalities (27) and (28) are satisfied. The equilibrium configuration (32) is reached, starting with any other configuration, asymptotically, as is readily seen by integrating equation (31). This gives

$$x = \frac{C_2}{C_1} + \frac{C_0}{C_1} e^{C_1 t},$$

where C_0 is a constant of integration. Since $C_1 < 0$ and $C_2 < 0$, the second term vanishes with increasing t, and x approaches the values C_2/C_1.

If neither inequality (27) nor (28) is satisfied, then $C_1 > 0$, $C_2 > 0$ and $dx/dt = 0$ for $x = C_2/C_1 > 0$. But $dx/dt > 0$ for $x > C_2/C_1$, and $dx/dt < 0$ for $x < C_2/C_1$. The configuration $x = C_2/C_1$ is unstable, and again either all passive individuals show behavior A or they all show behavior B.

An equation corresponding to (32) holds also for y. Hence,

$$\frac{x}{y} = \frac{a_0^* x_0 - c_0^* y_0 - (a - c_0^* \varepsilon' y_0) N'}{c_0^* y_0 - a_0^* x_0 - (a - a_0^* \varepsilon x_0) N'}, \tag{33}$$

which may be written

$$\frac{x}{y} = \frac{a_0^* \dfrac{x_0}{y_0} - [c_0^*(1 - \varepsilon' N') + \dfrac{a}{y_0} N']}{c_0^* - \dfrac{a}{y_0} N' - a_0^*(1 - N'\varepsilon) \dfrac{x_0}{y_0}}. \tag{34}$$

The quantity $1 - \varepsilon N'$ must be non-negative if equations (24) and (25) have physical meaning. Similarly, x and y must be non-negative. Since the denominator of the expression (32) for x and of the corresponding expression for y is positive because of inequality (28), therefore both numerator and denominator of equation (33) and hence of (34) are positive. It follows that

$$c_0^*(1 - \varepsilon' N') + \frac{a}{y_0} N' > 0 \text{ and } c_0^* - \frac{a}{y_0} N' > 0. \tag{35}$$

Hence, if we keep y_0 constant but increase x_0, so as to increase x_0/y_0, the ratio x/y will increase continuously, but will become infinite when

$$\frac{x_0}{y_0} = \frac{c_0^* - \dfrac{a}{y_0} N'}{a_0^*(1 - \varepsilon N')}. \tag{36}$$

For this and higher values of x_0/y_0, all passive individuals exhibit behavior A. On the other hand, when

$$\frac{x_0}{y_0} = \frac{c_0^*(1 - \varepsilon' N') + \dfrac{a}{y_0}}{a_0^*}, \tag{37}$$

then $x/y = 0$. For this and smaller values of x_0/y_0 all passive individuals exhibit behavior B.

It may seem strange that equations (36) and (37) are not symmetric. This is due to the fact that we vary x_0/y_0 for a fixed y_0. We obtain corresponding expressions if we keep x_0 constant.

Putting $a = 10$ ind./day, $a_0^* = c_0^* = 10^3$ ind./day, $N' = 10^7$ ind., $y_0 = 1{\cdot}3 \times 10^5$ ind., and $\varepsilon = \varepsilon' = 0{\cdot}8 \times 10^{-7}$, we find that $x/y = \infty$ when $x_0/y_0 = 1{\cdot}5$, although $x/y = 1$ when $x_0/y_0 = 1$. With x_0 and y_0 being only of the order of 1% of the total population, we see that a rather small change in the size of the active group may change the behavior of the population appreciably.

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CHAPTER IV

A MORE EXACT TREATMENT OF THE PREVIOUS CASE

It has been emphasized in the previous chapter that an assumption of a sharp division into active and passive individuals is made only as a convenient approximation. Actually, there are continuous gradations between the two groups. We shall now investigate a relatively simple case of such a gradation.

Generalization of the previously studied case is not unique. Different generalizations may lead to the same type of particular case. We must, therefore, choose more or less arbitrarily one of several possibilities without any prejudice to other possibilities to be studied later.[1]

Let us consider that every individual has a tendency to the activity A, measured by a coefficient $a < 1$. The quantity $1 - a$ measures, then, his tendency for the activity B. If $a = 1$ the individual's tendency for A is maximum and that for B is zero. Let the population be characterized by a distribution function $N(a)\,da$, giving the number of individuals having an a between a and $a + da$. We have

$$\int_0^1 N(a)\, da = N, \tag{1}$$

where N is the total population.

Denote by $x(a)\, da$ the number of individuals with a given a who exhibit behavior A. Denote by $y(a)\,da$ the number of individuals with a given a, exhibiting behavior B. We have

$$y(a) = N(a) - x(a). \tag{2}$$

In general, we must assume that the amount of influence that an individual exhibiting behavior A exerts towards an increase of behavior A in others is itself a function of a. An individual having $a = 1/2$ is completely indifferent as to the choice of behavior A or B. If he chooses a given behavior due to the influence of others, he will himself exert hardly any influence upon others to choose the same behavior. On the other hand an individual with $a = 0$ will not only choose of his own initiative behavior B, but will strongly influence others to choose that behavior. For simplicity, we shall choose for the amount of influence of an individual the expression

$$(1-2a)^2. \tag{3}$$

Expression (3) is everywhere positive, is zero for $a = 1/2$, and is also equal to 1 for $a = 0$ or for $a = 1$. Accordingly, we consider that an individual, characterized by a given a and exhibiting behavior A, influences other individuals to choose A to the extent

$$f_A(a) = a(1-2a)^2, \tag{4}$$

while an individual with the same a, but exhibiting behavior B, influences other individuals to choose B to the extent

$$f_B(a) = (1-a)(1-2a)^2. \tag{5}$$

Thus an individual with $a = 1/2$ does not exert any influence one way or another, as stated before. An individual with $a = 1$, when performing activity A, influences others to perform A in the amount a. An individual with $a = 1$, but performing activity B, does not influence others to perform B at all. This is psychologically plausible. For an individual with $a = 1$ can perform B only under duress, since it is contrary to his inclinations, and he certainly will not attempt to induce others to do the same thing.

As stated above, individuals with $a = 1$ always exhibit behavior A regardless of what others do. Since, mathematically speaking, the number of such individuals $N(1)da$ is infinitesimal, even if $N(1)$ is large, we must consider that all individuals whose a lies between 1 and $1 - \Delta$, where Δ is a small quantity, always exhibit behavior A, regardless of the behavior of others. Similarly, all individuals with a lying between 0 and Δ always exhibit behavior B. The quantities

$$\int_{1-\Delta}^{1} x(a)\,da \quad \text{and} \quad \int_{0}^{\Delta} [N(a) - x(a)]\,da \tag{6}$$

play the roles of x_0 and y_0 of the previous chapters, respectively.

With these assumptions we may set, by an argument similar to that used on page 27 of chapter iii and with a' as the integration variable:

$$\begin{aligned} \frac{dx(a)}{dt} = &\int_0^1 a'(1-2a')^2\,x(a')\,da' \\ &- \int_0^1 (1-a')(1-2a')^2\,[N(a') - x(a')]\,da'. \end{aligned} \tag{7}$$

Since the integrals are constants, independent of a, putting

$$X = \int_0^1 x(a)\, da\,, \tag{8}$$

we have

$$\frac{dX}{dt} = \int_0^1 a'(1-2a')^2 x(a')\, da' - \int_1^1 (1-a')(1-2a')^2 [N(a') - x(a')]\, da'\,. \tag{9}$$

At all times, however, we have

$$\begin{aligned} x(a') &= N(a') \quad \text{for} \quad 1-\Delta < a' < 1\,; \\ x(a') &= 0 \quad\quad\;\; \text{for} \quad 0 < a' < 1\,. \end{aligned} \tag{10}$$

Everywhere else $x(a')$ is determined only by the initial conditions.

If $N(a)$ is symmetrical with respect to $a = 1/2$, so that $N(a) = N(1-a)$, then if for $t = 0$, $x(a') = 0$ everywhere in the interval $[0, (1-\Delta)]$, the first integral of equation (9) is less than the second for $t = 0$. The first integral is then equal to

$$k_1 = \int_{1-\Delta}^1 a'(1-2a')^2 N(a')\, da'\,. \tag{11}$$

while the second may be written as

$$k_2 = \int_0^\Delta (1-a')(1-2a')^2 N(a')\, da' + \int_\Delta^{1-\Delta} (1-a')(1-2a')^2 N(a')\, da'\,. \tag{12}$$

Because of the symmetry of $N(a')$, k_1 is equal to the first integral of expression (12), the second being positive. Hence $k_1 < k_2$. In that case, according to equations (7) and (9), X can only decrease for all $a < 1 - \Delta$. Hence, all individuals with $a < 1 - \Delta$ will continually exhibit behavior B. Similarly for a symmetric $N(a)$, if at the beginning $x(a') = N(a')$ everywhere in the interval $(\Delta, 1)$, we find that such a situation remains unchanged. Hence, as in chapter iii, for a symmetric $N(a)$ (corresponding to $x_0 = y_0$, $a_0 = c_0$), we have two possible configurations, either a behavior A by almost the whole group, or behavior B.

The following considerations emphasize still more the analogy with the former results. Since equation (7) is independent of a in the right member, $x(a, t)$ consists of two components

$$x = x_1(a) + x_2(t)$$

where the first component is independent of t and the second of a. If

$$I_1 = \int_0^1 (1-2a)^2 (1-a) N(a)\, da,$$

$$I_2 = \int_0^1 a(1-2a)^2 x_1(a)\, da,$$

the latter being a functional $I_2(x_1)$, and

$$I = I_2 - I_1,$$

then I is independent of a and of t, though I is also a functional of x_1. Equation (7) now becomes

$$\frac{dx_2}{dt} = \frac{1}{3} x_2 + I.$$

It is no restriction to suppose that $x_2(0) = 0$ and hence that $x_1(a)$ gives the initial distribution. Hence,

$$x_2 = 3I(e^{t/3} - 1). \tag{13}$$

The increase is exponential and is in favor of A when $I > 0$, in favor of B when $I < 0$.

Equation (13) is analogous to equation (21) of chapter iii.

If $N(a)$ is asymmetric, then the whole situation may change. In that case k_1 is not necessarily smaller than k_2. Let the asymmetry favor large a's, so that $N(a) < N(1-a)$ for $a < 1/2$. Denote the two integrals of equation (12) by k_3 and k_4 respectively, so that

$$k_2 = k_3 + k_4. \tag{14}$$

We now have

$$k_1 > k_3.$$

If

$$k_1 - k_2 > k_4, \tag{15}$$

then

$$k_1 > k_2, \tag{16}$$

and $x(a)$ will increase everywhere, except for $a < \Delta$. But an increase of $x(a)$ reduces k_2 and increases k_1, thus further enhancing inequality (16). Hence, the increase of $x(a)$ will continue until all individuals, except those with $a < \Delta$, exhibit behavior A. Thus inequality (15) is the condition for the group to pass from behavior B into behavior A. Condition (15) may actually require a very small asymmetry if $N(a)$ is large only in the immediate neighborhood of $a = 1/2$, where $(1 - 2a')^2$ is very small, for in that case k_4 is a

small quantity, and a very slight asymmetry of $N(a)$ will result in inequality (15). But there is always a threshold value for the necessary asymmetry.

By a similar argument, we find that an asymmetry of $N(a)$ in favor of smaller a's, if exceeding a threshold value, will result in behavior B for the whole group, except for individuals with $a > 1 - \Delta$. Those results are essentially identical with the results of the more restricted treatment of chapter iii.

More complicated relations may be studied by considering the case of different susceptibility of the different individuals to the influence of others. We may, for instance, consider that the susceptibility of an individual to the influence of others exhibiting behavior A is proportional to the value of a of that individual, while his susceptibility to the influence of others exhibiting behavior B is proportional to $1 - a$. We shall then have, instead of equation (7),

$$\frac{dx(a)}{dt} = a \int_0^1 a'(1-2a')^2 x(a')\, da' - (1-a) \int_0^1 (1-a')(1-2a')^2 [N(a') - x(a')]\, da'. \tag{17}$$

Investigations of this and other more complex cases present an interesting study.

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CHAPTER V

ECONOMIC INTERACTION IN A SOCIAL GROUP

In chapter iii we discussed some possible interrelations between social classes based on the different amounts of initiative and imitativeness. In this chapter we shall discuss an interaction of classes based on a different characteristic, namely, the difference in different people of the ability to perform different kinds of work and to produce different goods. In a sense, such a discussion should lead us into the domain of mathematical economics, and in this field a very considerable progress already has been made by well known works[1] of mathematical economists. However, we shall approach these problems from a somewhat different point of view which will bring out particularly the social aspect of the interrelation of economic classes. In order to do so, we shall start again with highly oversimplified hypothetical cases. In view of the fact that much more complex and realistic cases have already been treated, it may seem rather strange to start again with oversimplified cases. However, as we shall see, the problem which interests us here is different from the usual type of problem treated, offering its own complications; this consideration justifies the apparent set back. Our failure to make use of those well-known results is due not to ignorance or neglect, but to the circumstance that the particular problems discussed here are not yet ripe for the use of those more advanced results.

Let us again consider for simplicity a society composed of two types of individuals I and II, their corresponding numbers being N_1 and N_2. Let these two types of individuals form two corresponding social classes, class I and class II. Let us consider the following purely hypothetical case.[2]

Each individual of type I produces per unit time an amount p_1 of goods necessary for the maintenance of his life, and he consumes per unit time an amount c_1 of these goods. Similarly every individual of type II produces and consumes correspondingly the amounts p_2 and c_2. Let

$$N_1 << N_2; \quad \varepsilon_1 = p_1 - c_1 > 0 \quad \text{and} \quad \varepsilon_2 = p_2 - c_2 < 0 . \tag{1}$$

Then, in the absence of any exchange of external supply, the indi-

viduals of type I are capable of indefinite existence, those of type II are not.

Now let the individuals of type I be capable of *organizing* their work, as well as the work of individuals of type II, in such a way that the production of the necessary goods is largely increased. In other words, let the production of the goods require a certain amount of *organization skill*, which is possessed by individuals of type I, but not by those of type II, so that while by themselves the individuals of type II cannot produce enough goods, they can do so under the direction of the individuals of type I.

Although this is a rather abstract situation, it reminds one of some real situations. Thus a large number of workers who produce complicated machinery in a factory as a result of work directed by a group of engineers and executives would not be able to produce the same results when left to themselves.

Under the conditions assumed above, the following relation may naturally be established between the individuals of type I and those of type II. The latter will agree to work under the direction of the former so as to increase their output in goods, provided they receive a part of the goods thus produced for their consumption. On the other hand, the individuals of type I, unless they act purely altruistically, a supposition too unlikely to be considered even theoretically, will agree to direct the work of type II individuals only if they also receive part of the goods thus produced. Each individual of type II will give an amount w of his labor per unit time. The amount w may be conveniently measured in working hours per day. In compensation for that amount of work, he will get an amount θw of goods. The quantity θ may be considered as the price of labor, paid not in money, but in goods directly.

If, for simplicity again, we consider the case in which all N_2 individuals of type II are working in the way discussed above, the total amount P of goods produced *under the direction* of individuals of type I will be proportional to $N_2 w$; thus

$$P = AN_2 w\,. \tag{2}$$

The coefficient of proportionality A will in general depend on both N_1 and N_2, so that

$$A = A(N_1, N_2)\,. \tag{3}$$

The nature of this functional dependence will be determined by the character of the labor involved, methods of production, etc. In the simplest case, however, it can be seen that $A(N_1, N_2)$ should be, at least approximately, a function of N_1/N_2, so that

$$A = f\left(\frac{N_1}{N_2}\right) > 0. \tag{4}$$

The reason for this assertion will be apparent if we remember that, other conditions being kept constant, increasing m times both N_1 and N_2 should increase P approximately m times also. Hence, such an increase must leave A invariant.

Some other properties of the function f in inequality (4) may be found from general considerations. Since in the absence of individuals of type I no goods are produced *under their direction*, we must have

$$f(0) = 0. \tag{5}$$

Keeping N_2, the number of workers available, constant but increasing the ratio

$$\eta = \frac{N_1}{N_2} \tag{6}$$

by increasing N_1, will, for small values of N_1, increase P. But when for a constant N_2, the number N_1 exceeds a certain amount, any further increase of N_1 will not result in any increase of P. A certain number of workers may require a definite number of supervisors, but an excessive number of supervisors does not increase the output. Hence, $f(\eta)$ must either have a maximum for a certain value of η, or it must increase with η, approaching asymptotically a constant value.

In order to obtain closed expressions, we may consider as a very rough approximation the following expression:

$$f(\eta) = f_0(1 - e^{-a\eta}). \tag{7}$$

This, of course, does not preclude considerations of other forms. We may take also

$$f(\eta) = a_1\eta - a_2\eta^2, \tag{8}$$

and consider this expression only in the range $0 < \eta < a_1/a_2$, for which the expression (8) is positive. It has a maximum for $\eta = \frac{1}{2}(a_1/a_2)$, the maximum value being $f_m = \frac{1}{4}(a_1^2/a_2)$.

Introducing equation (6) into equation (4) and then the latter into (2), we have

$$P = f(\eta)\, N_2 w. \tag{9}$$

Of the total amount P of goods produced, an amount $\theta N_2 w$ is received by the N_2 individuals of type II, the remaining amount being retained by those of type I. Hence, the net rate of increase of the total amount W_1 of goods, produced by individuals of type I themselves and by those of type II for class I, is:

$$\frac{dW_1}{dt} = N_1\varepsilon_1 + f(\eta)N_2w - \theta N_2w; \tag{10}$$

and, similarly for class *II*:

$$\frac{dW_2}{dt} = N_2\varepsilon_2 + \theta N_2w. \tag{11}$$

Even though, according to expressions (1) $\varepsilon_2 < 0$, dW_2/dt may be positive, since $\theta N_2 w > 0$. It must be remarked, however, that we obtain the same type of interaction of the two classes even when both ε_1 and ε_2 are positive. This type of cooperative interaction may increase the rates dW_1/dt and dW_2/dt from their values $N_1\varepsilon_1$ and $N_2\varepsilon_2$ which they would have in the absence of such a cooperation. This presupposes such a choice of constants that

$$f(\eta) > \theta. \tag{12}$$

Inequality (12) fixes the highest "price" that the first class will be willing to pay for labor. The actual θ is determined by the demand of the individuals of type *II* for the goods produced. Just as in the standard treatments of mathematical economics, the larger θ per unit of w, the more of w an individual is willing to give. Hence

$$w = u(\theta), \tag{13}$$

u being an increasing function of θ. For simplicity let us put

$$w = w_0 - \frac{b}{\theta}, \tag{14}$$

w_0 and b being constants. Expression (14) is suggested by the use of simple linear approximations for demand curves and by remembering that $\frac{1}{\theta}$ may be considered as the "price" of the goods in terms of labor. For $\theta < b/w_0$, when w becomes negative, individuals of type *II* will not agree to work at all. For $\theta = \infty$ they cannot work more than the physically possible amount w_0.

Introducing equation (14) into (10) gives:

$$\frac{dW_1}{dt} = N_1\varepsilon_1 + N_2[f(\eta) - \theta]\left(w_0 - \frac{b}{\theta}\right). \tag{15}$$

The derivative dW_1/dt is now a function of θ, and if the individuals of type *I* wish to make dW_1/dt as large as possible, they will agree on a θ which makes the right-hand side of equation (15) a maximum. This happens for

$$\theta = \sqrt{\frac{bf(\eta)}{w_0}}, \tag{16}$$

and, for the maximum value of the right-hand side of equation (15), the latter becomes

$$\frac{dW_1}{dt} = N_1\varepsilon_1 + N_2[w_0 f(\eta) + b - 2\sqrt{bw_0 f(\eta)}]. \tag{17}$$

Introducing equation (16) into (14), and then both into expression (11), we find:

$$\frac{dW_2}{dt} = N_2[\varepsilon_2 - b + \sqrt{bw_0 f(\eta)}]. \tag{18}$$

Equation (17) may be written thus:

$$\frac{dW_1}{dt} = N_2[\varepsilon_1\eta + w_0 f(\eta) + b - 2\sqrt{bw_0 f(\eta)}]. \tag{19}$$

In order for any such interaction as discussed here to be possible at all, both classes must gain something. This imposes certain inequalities on the constants, namely,

$$\varepsilon_1\eta + w_0 f(\eta) + b - 2\sqrt{bw_0 f(\eta)} > 0, \tag{20}$$

and

$$\varepsilon_2 - b + \sqrt{bw_0 f(\eta)} > 0. \tag{21}$$

If inequality (21) is to hold, and if $\varepsilon_2 < 0$, then a necessary condition is

$$\sqrt{bw_0 f(\eta)} - b > 0.$$

Because of equation (16), and since necessarily $\theta > 0$, this requires that

$$b < w_0 f(\eta), \tag{22}$$

but inequality (22) is not sufficient in this case. In the interval $0 < b < w_0 f(\eta)$, the expression $\sqrt{bw_0 f(\eta)} - b$ is positive and has a maximum for

$$b = b_m = \frac{w_0 f(\eta)}{4}, \tag{23}$$

the maximum value being equal to

$$\frac{w_0 f(\eta)}{4}. \tag{24}$$

Hence, for $\varepsilon_2 < 0$, inequality (21) requires not only inequality (22) but also

$$\frac{w_0 f(\eta)}{4} > |\varepsilon_2|. \tag{25}$$

Requirement (20) may be replaced by a more rigid one, namely,

$$w_0 f(\eta) + b - 2\sqrt{b w_0 f(\eta)} > 0. \tag{26}$$

Only in this case is dW_1/dt larger with the cooperative arrangement than without it. But for values of b in the neighborhood of b_m the expression $b - 2\sqrt{b w_0 f(\eta)}$ is negative. However, for $b = b_m$ the left-hand side of inequality (26) equals $\frac{1}{4} w_0 f(\eta)$, and (26) is therefore satisfied.

These considerations show that b would have to be in the neighborhood of b_m in order that there should be a gain from cooperation to individuals of class II if $\varepsilon_2 < 0$. A value of b too large or too small results in a loss to the individuals of type II. If, however, $\varepsilon_2 > 0$, then all that is necessary is that b should be sufficiently small. In this case too large a "price" asked for labor will result in the impossibility of reaching any agreement or cooperation.

If $\varepsilon_2 < 0$ and inequality (25) holds, then with $b \sim b_m$, $dW_1/dt > dW_2/dt$. Equations (19) and (18) integrated give:

$$W_1 = N_2[\varepsilon_1 \eta + w_0 f(\eta) + b - 2\sqrt{b w_0 f(\eta)}]\, t + W_{01}, \tag{27}$$

$$W_2 = N_2[\varepsilon_2 - b + \sqrt{b w_0 f(\eta)}]\, t + W_{02}, \tag{28}$$

where W_{01} and W_{02} are initial values. If we start with equal amounts of W per capita, so that

$$\frac{W_{01}}{W_{02}} = \frac{N_1}{N_2}, \tag{29}$$

or

$$W_{01} = \eta W_{02}, \tag{30}$$

then for $t = 0$, $W_1/W_2 = \eta$, while for sufficiently large t this ratio tends to

$$\frac{W_1}{W_2} = \frac{\varepsilon_1 \eta + w_0 f(\eta) + b - 2\sqrt{b w_0 f(\eta)}}{\varepsilon_2 - b + \sqrt{b w_0 f(\eta)}} > 1 > \eta. \tag{31}$$

Thus, there will be a gradual increase of the ratio of the amount of goods accumulated by class I to the amount accumulated by class II.

We may consider a somewhat more complex and perhaps more

natural situation in which an individual of type *II* receives goods not only in proportion to the amount of labor he expends per unit time, but also in proportion to the amount of goods which he produces per unit time under the direction of individuals of type *I*.[3]

Expression (2) tacitly implies that the amount of supervisory work done by individuals of type *I* per unit time is constant. It is of interest to consider the more general case, in which P is proportional to the product $w_1 w_2$ of the amounts of labor w_1 and w_2 given per unit time by individuals of type *I* and *II*. The quantity P is by definition the amount of goods produced per unit time by individuals of type *II* *under the direction of individuals of type I*. Therefore, P should vanish for either $w_1 = 0$ or $w_2 = 0$. Thus, we shall now consider the case where

$$P = f\left(\frac{N_1}{N_2}\right) N_2 w_1 w_2 , \tag{32}$$

$f(\eta)$ being the same as before.

We now find, instead of equations (10) and (11), the following equations for the rates of change of the total amount of goods W_1 and W_2 possessed by all individuals of type *I* and *II* correspondingly:

$$\frac{dW_1}{dt} = N_1 \varepsilon_1 + N_2 f(\eta) w_1 w_2 (1 - \theta) , \tag{33}$$

$$\frac{dW_2}{dt} = N_2 \varepsilon_2 + \theta N_2 f(\eta) w_1 w_2 . \tag{34}$$

By definition of θ we always have $\theta < 1$.

We have supposed that the 'price" θ is related to the amount of labor w_2 by a simple hypothetical demand equation:

$$w_2 = w_{02} - \frac{b_2}{\theta} \tag{35}$$

where w_{02} and b_2 are constants. A similar situation may be considered for w_1. Individuals of type *I* will be willing to give more supervisory labor, the greater the amount of additional goods they receive in return. We shall, however, introduce here in our assumptions an explicit asymmetry in the behavior of the two classes. This is made partly in order to simplify some calculations. The effects of *not introducing* such an asymmetry will be discussed at the end of this chapter.

We shall assume that while the individuals of class *II* determine the amount of labor they are willing to give by *the fraction* of the goods which they receive in return per unit of goods produced, the in-

dividuals of class I determine the amount of supervisory labor which they are willing to give by the *total* amount of goods which they receive from class II per individual of class I as the result of cooperation. Thus we shall put

$$w_1 = w_{01} - \frac{b_1}{P(1-\theta)/N_1}.$$

Introducing for P its expression (32), we find:

$$w_1 = w_{01} - \frac{b_1 \eta}{f(\eta) w_1 w_2 (1-\theta)}. \tag{36}$$

Equation (36) may be written:

$$f(\eta)(1-\theta) w_2 w_1^2 - w_{01} f(\eta)(1-\theta) w_2 w_1 + b_1 \eta = 0. \tag{37}$$

Solved with respect to w_1, it gives:

$$w_1 = \frac{w_{01}}{2} \pm \frac{\sqrt{w^2_{01} f^2(\eta)(1-\theta)^2 w^2_2 - 4 b_1 \eta f(\eta)(1-\theta) w_2}}{2 f(\eta)(1-\theta) w_2}. \tag{38}$$

We shall now discuss the second term on the right side of equation (33). This term represents the amount of goods received per unit time by all individuals of type I from those of type II. To this end consider the expression $w_1 w_2 (1-\theta)$. From equations (38) and (35) we find after rearrangements:

$$w_1 w_2 (1-\theta) = \frac{w_{01}}{2}\left(1-\theta\right)\left(w_{02} - \frac{b_2}{\theta}\right) \times \left[1 \pm \sqrt{1 - \frac{4 b_1 \eta}{f(\eta)(1-\theta) w^2_{01}(w_{02} - b_2/\theta)}}\right]. \tag{39}$$

The expression

$$x = (1-\theta)(w_{02} - b_2/\theta), \tag{40}$$

which occurs in the denominator under the radical sign, has a maximum for

$$\theta_m = \sqrt{\frac{b_2}{w_{02}}}. \tag{41}$$

The maximum value x_{max} of x is equal to:

$$x_{max} = w_{02} + b_2 - 2\sqrt{b_2 w_{02}}. \tag{42}$$

If

$$\frac{4b_1\eta}{f(\eta)\,w^2_{01}x_{max}} < 1\,,$$

or

$$x_{max} > \frac{4b_1\eta}{f(\eta)\,w^2_{01}}\,, \tag{43}$$

then the expression $w_1w_2(1-\theta)$ has two real positive values for a given θ, provided θ lies within an interval

$$\theta_1 < \theta < \theta_2 < 1\,, \tag{44}$$

which is in the neighborhood of the value given by equation (41). The quantities θ_1 and θ_2 are roots of the equation obtained by equating the right-hand side of equation (40) to the right-hand side of inequality (43).

If inequality (43) does not hold, then $w_1w_2(1-\theta)$, and therefore the last term of equation (33) is imaginary. Under these conditions a cooperation of the two groups is impossible and no agreement as to price can be reached. It is interesting to note that inequality (43), which represents the conditions for the possibility of cooperation, involves such quantities as the function $f(\eta)$ or the constant w_{01}, which are more or less of a physical nature, as well as the constant b_1, which is more of a psychological nature.

The existence of two values of the expression $w_1w_2(1-\theta)$ is due to a two-valued nature of w_1, as defined by equation (36). This, however, is not connected with the specific analytical expression chosen for w_1, but lies rather in the nature of the general assumption made. The relation of w_1 to the return in goods is such that for returns below a certain value the individual will not wish to give any labor, while for increasing return the amount of labor invested increases, tending to a limiting value determined by the physically possible maximum of labor given. This, combined with the assumption that the return in goods is in its turn proportional to the amount of labor given, leads to the two-valued character of w_1 in terms of θ. The left-hand side of equation (36) is represented by a straight line, (Figure 1), while the right-hand side is represented by a curve which, regardless of its detailed analytic specification, *in general,* intersects the straight line in two points.

Returning now to equation (39), we see that the value of $w_1w_2(1-\theta)$ which corresponds to the upper sign before the radical is always the largest of the two. When $\theta=\theta_m$, as given by equation (41), both the expression in brackets and the expression in front of it have a maximum. Hence, for $\theta=\theta_m$, the quantity $w_1w_2(1-\theta)$ has

a maximum for the upper sign before the radical. For the lower sign it may or may not have a maximum.

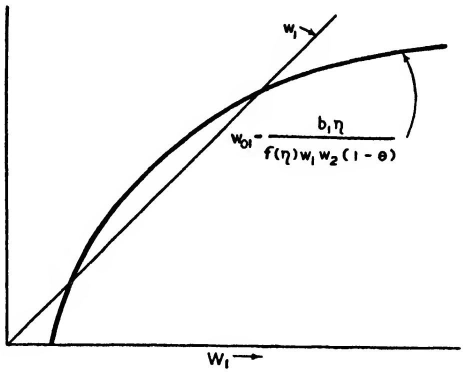

FIGURE 1

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In chapter iii we have established some equations describing the conditions under which one of the two classes controls the behavior of the other regardless of such interactions as studied here. If the relations between the constants involving social control are such that in the present case class I, which is composed of individuals of type I, controls behavior of class II (an assumption which is not *a priori* necessary), then class I will prescribe as price for the labor of class II the value θ_m, for this makes dW_1/dt a maximum.

Due to inequalities (1), dW_1/dt is always positive for $\theta < 1$, but dW_2/dt may be negative. If $\theta = \theta_m$, then, in order to make dW_2/dt positive also, we must have

$$\theta w_1 w_2 > -\frac{\varepsilon_2}{f(\eta)} \quad \text{for } \theta = \theta_m ; \tag{45}$$

otherwise, the "wealth" of class I will increase while that of class II will decrease. Class II will not be able to exist indefinitely, and without it class I will also not be able to have additional goods. A continued negativity of dW_2/dt will result in a decrease of N_2, and this will result in a decrease of P and of the last term of equation (33). Thus condition (44) is necessary, but not sufficient, for the coexistence of two such classes. It must be supplemented by condition (45).

If class I fixes θ at θ_m, this will make dW_1/dt a maximum, but this does not mean that the rate of increase of the total goods for both classes, that is $d(W_1 + W_2)/dt$, will be a maximum. In fact, this will

not be the case. From equations (33) and (34) it follows that $d(W_1 + W_2)/dt$ has a maximum for such a value θ'_m of θ, for which w_1w_2 has a maximum. Denote

$$y_1 = w_1w_2 > 0; \quad y_2 = w_1w_2(1-\theta) = y_1(1-\theta) > 0. \tag{46}$$

The quantity θ'_m is defined as that value of θ which makes y_1 a maximum. Hence

$$\frac{dy_1}{d\theta} = 0 \quad \text{for} \quad \theta = \theta'_m. \tag{47}$$

If θ_m is the value which makes $y_2 = y_1(1-\theta)$ a maximum, then

$$\frac{dy_2}{d\theta} = 0 \quad \text{for} \quad \theta = \theta_m.$$

But for $\theta = \theta'_m$, because of equation (47),

$$\frac{dy_2}{d\theta} = -y_1(\theta'_m) + (1-\theta'_m)\frac{dy_1}{d\theta} = -y_2 < 0\,. \tag{48}$$

Hence $\theta'_m > \theta_m$.

In other words, under such conditions as are described by equations (33), (34), (35) and (36), such a cooperation of the two classes which is optimal for the population as a whole will not be optimal for class I.

Denoting by θ''_m the value of θ which makes $\theta w_1 w_2$ a maximum and therefore gives a maximum dW_2/dt, we can prove by a similar argument that

$$\theta''_m > \theta'_m > \theta_m\,. \tag{49}$$

If class II controls the behavior of the population, or if a third class is present which, while not participating directly in the cooperation discussed here, nevertheless influences the behavior of the population by virtue of a social mechanism discussed in chapter iii, then θ may be fixed either at θ'_m or at θ''_m. This will lead to optimal conditions for either the whole population or for class II. It may, however, happen that $\theta''_m > \theta_2$, or even $\theta'_m > \theta_2$ [Equation (44)]. In this case no cooperation will be possible at all. Class I will continue to exist due to $\varepsilon_1 > 0$, though the rate of increase of its wealth will be very much reduced. Class II will not be able to exist indefinitely. Under the conditions studied here, the coexistence of the two classes may be possible for a certain choice of constants, only when it corresponds to conditions close to those optimal for class I.

If we do not introduce the asymmetry expressed in the different structure of equations (35) and (36), but take for w_1 an expression

similar to (35) except that θ is replaced by $(1-\theta)$, we will have simplified the situation somewhat. We now would have:

$$(1-\theta)w_1w_2 = (1-\theta)\left(w_{01} - \frac{b_1}{1-\theta}\right)\left(w_{02} - \frac{b_2}{\theta}\right) . \tag{50}$$

which has a maximum for

$$\theta = \theta_m = \sqrt{\frac{(w_{01} - b_1)b_2}{w_{01}w_{02}}}. \tag{51}$$

As before, we shall find $\theta''_m > \theta'_m > \theta_m$.

On the other hand, if we take for both w_1 and w_2 expressions of the form of equation (36), then the situation is mathematically more complex. Although essentially similar results are obtained, closed expressions are very cumbersome, for we have to deal now with two simultaneous quadratic equations which reduce to a single fourth degree equation.

The different expressions that may be used for w_1 and w_2 correspond to different assumptions about the psychological attitudes taken by each class towards cooperation of the type described. The relations discussed here are too simple to be of any practical use. They merely *illustrate* how different psychological attitudes of different social classes may be in principle translated into mathematical language. But it is only through the study of such simple *illustrations* that we may hope to arrive at the study of more concrete real cases.

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CHAPTER VI

MORE COMPLEX CASES OF INTERACTION OF SOCIAL CLASSES

I

The different types of interaction of social classes studied in chapter iii–v are substantially of two kinds. Those treated in chapter iii and iv may be described as more of a "psychological" nature; those discussed in chapter v may be said to be primarily of an "economic" character. No sharp distinction can be drawn, however, between the two kinds. The two different activities A and B discussed in chapter iii may, for instance, consist of the buying and use of certain goods or services, and the two opposing active groups may represent two competing groups of manufacturers trying to impose by advertising the use of their corresponding products on the passive population.

The individuals composing the "organizing" class, discussed in chapter v, need not necessarily be "active" in the sense defined in chapter iii. A person may have organizing and technical abilities but not care particularly to use those abilities. If, however, the exhibition of such abilities is considered desirable by a larger group of people, the person may actually put them to use. Thus, he may be a sort of "passive technical organizer."

In spite of this lack of sharp distinction, the two kinds of interactions may perhaps typify some actually existing situations. In any country we distinguish, in general, two groups of individuals who characterize the general sociological setup of the country.

One class supplies the legislative activity, sets forth the legal codes to be followed, and sees to the enforcement of the laws. This class is not infrequently also the military class, in which a control of a large number of passive individuals by a smaller number of active ones is certainly *not* based on any economic interaction.

The other class is composed of industrialists and tradesmen—people that are primarily interested in the process of producing different goods for the improvement of conditions of life and thus constitute the economic backbone of the country.

Again it must be said that sometimes there are no sharply indicated lines of demarcation between the two classes. In extreme cases,

however, the possibility of such a division becomes apparent. We have on one hand highly militaristic countries, such as tsarist Russia or prewar Japan, with a relatively small industrial production and small per capita national income*; on the other hand, we have such countries as the United States and Canada with high national per capita incomes but with very little, if any, military activities.

It lies essentially in the nature of any military or legislative activity that it must be imposed on the population by decree rather than by mutual agreement. On the other hand, the "economic" interaction studied in chapter v is based fundamentally on mutual agreement apparently free from overt compulsion. We should therefore expect a relatively high coefficient of correlation between technical organizing abilities and general tolerance to the opinion of others, as well as a higher interest in the worldly goods and comforts of life. We should also expect a relatively high correlation between the military and legislative abilities and general autocratic tastes, as well as a lesser interest in the material comforts of life. There may be some parallelism between the subdivision into these two types and P. Sorokin's[1] division of individuals into the ideational and sensate.

This parallelism is not complete, however, primarily because we should consider still another type of activity, which is better described by the interaction studied in chapter iii, and yet is not of a military or legislative nature. This type includes religious activities which, as history shows, have been frequently imposed on the population by methods very far from those using free mutual agreement. An extreme case of a country with a strongly pronounced active group of that type would lack both industrial and military development. China and India may perhaps be considered as approximately that type.

Moreover, a country may possess a sufficiently strong "organizing" class to provide for high technical developments; yet the military class may be even stronger and impose its general behavior on the population. We may thus have a country with high technical advances, yet characterized by a strong political intolerance. Such was the case, for example, with Nazi Germany.

A class may also be subdivided into subclasses, the interaction of which may be studied. Thus the technical and organizing class may be divided into the industrial and agricultural class, etc.

The above discussion indicates the desirability of a theoretical study of interaction between two or more classes, of which some pos-

* Although the per capita income in Soviet Russia is also rather low, we refer to tsarist Russia only because Soviet Russia appears to be in a transient period of rapid industrial expansion.

sess the characteristics described in chapter iii, some—the characteristics described in chapter v. For brevity we shall call the former the "autocratic" class, the latter the "organizing" class. We shall confine ourselves here to the study of three classes: an autocratic, an organizing and a passive one.

To begin with, we may consider the following case[2]: Let the number of individuals in corresponding classes be N_1, N_2, and N_3. Let

$$N = N_1 + N_2 + N_3 . \tag{1}$$

We shall denote the corresponding coefficients of influence by a_1, a_2, and a_3. Let us consider the first type of interaction, that of constant effort. Let

$$N_1 > \frac{a_3}{a_1 + a_3} N + \frac{a_2 - a_1}{a_1 + a_3} N_2 . \tag{2}$$

This is essentially the inequality (8) of chapter iii, with changed notations. It expresses the condition for class I to control the behavior of the whole passive population. The general behavior pattern will be determined by the dictates of class I.

Let class II, on the other hand, be an organizing class. If the organization of class III by class II, in order to increase the production of some useful goods, will not interfere with the general behavior pattern imposed by class I, then class II will be permitted to proceed with such an organization, provided a certain amount of the useful goods produced will be given to class I.

Concerning the interaction between class II and class III, several different assumptions may be made as we have seen. For the sake of definiteness, we shall consider here, *as an illustration of the method only* and without any prejudice in favor of it, the case discussed on page 46. Again denoting by θ the fraction of goods produced which is given to class III, by w_2 and w_3 the amounts of labor given by class II and III respectively, and assuming as before "demand functions" of the form

$$w_2 = w_{02} - \frac{b_2}{1 - \theta}, \quad w_3 = w_{03} - \frac{b_3}{\theta}, \tag{3}$$

we find that the amount of goods received by class II, in the absence of any interference by class I, is

$$N_3 f(\eta)(1 - \theta)\left(w_{02} - \frac{b_2}{1 - \theta}\right)\left(w_{03} - \frac{b_3}{\theta}\right). \tag{4}$$

Let us consider, however, that class I, which controls the whole situation, will require a certain fraction α of everything produced by the cooperation of classes II and III. Thus class II will now retain only a fraction $(1-\theta)(1-\alpha)$ of goods, while class III retains only $\theta(1-\alpha)$. Those amounts should be introduced now into expression (4), instead of $(1-\theta)$ and θ, respectively. Moreover, it may be argued that the constants b_2 and b_3 are functions of f, for f measures the efficiency of supervision, and the greater f, the larger the total amount produced. Thus the greater f, the smaller the fraction of the goods produced which may be worth retaining. Hence we put

$$b_2 = \frac{b_2'}{f}, \quad b_3 = \frac{b_3'}{f}. \tag{5}$$

Instead of expression (4) we now have

$$N_3 f(1-\theta)\left(w_{02} - \frac{b_2'}{f(1-\theta)(1-\alpha)}\right)\left(w_{03} - \frac{b_3'}{f\theta(1-\alpha)}\right) \tag{6}.$$

Class II will fix θ in such a way as to make this quantity a maximum. For a fixed α, the value θ_m of θ which maximizes (6) is equal to

$$\theta_m = \sqrt{\frac{\left(w_{02} - \dfrac{b_2'}{f(1-\alpha)}\right)\dfrac{b_3'}{f(1-\alpha)}}{w_{02}\, w_{03}}}. \tag{7}$$

On the other hand, class I, if it controls the whole population completely, will fix α at a value α_m in such a way as to maximize the total amount it receives, namely,

$$\alpha N_3 f(1-\theta)\left(w_{02} - \frac{b_2'}{f(1-\theta)(1-\alpha)}\right)\left(w_{03} - \frac{b_3'}{f\theta(1-\alpha)}\right). \tag{8}$$

By substituting (7) into (8), we obtain the latter expression as a function of α only. By differentiating we then find the value of α_m in terms of N_3, f, w_{02}, w_{03}, b_2' and b_3'. Substituting that value α_m into (7) we find θ_m as a function of the foregoing six quantities. Once θ_m and α_m are determined, we have the total rate of accumulation of goods by each of the three classes, according to equations set up in chapter v.

Relation (4), chosen as an illustration, does not allow any expression for θ_m and α_m in closed form. The same holds about other similar relations. More or less complicated approximate solutions of this problem are hardly worth while at this stage since the assumptions made are too crude to be applied to actual cases and since hardly any accurate data on the subject are available. As a second illustra-

tion we shall consider a somewhat different picture, which will be more familiar to the student of mathematical economics.

Consider class *II* as a monopolist, producing an amount u of goods at a cost $q(u)$, and let[3]

$$q(u) = Au^2 + Bu + C\,, \quad (A\,, B\,, C > 0)\,. \tag{9}$$

Let the goods be sold at a price p per unit, and let there be a demand function for the goods of the form

$$y = b' - a'p\,, \quad a', b' > 0\,. \tag{10}$$

In actual cases $q(u)$ is composed of the cost of labor as well as cost of material, transportation, etc. We shall consider here, for simplicity, the fictitious case where $q(u)$ consists only of the cost of labor. The situation, though unreal, is not quite impossible. If we consider the interaction of several industries, then there is an exchange of materials between them, which enters into $q(u)$. The cost of material for one industry amounts ultimately, however, to the cost of labor in another industry, which produces it. Therefore if we consider a sort of *average* for all industries, and denote by u the amount of goods produced by all of them, then q eventually becomes the cost of labor only. This, of course, introduces the question of whether we can measure the products of different industries in common units. Inasmuch as we are discussing here only a theoretical case, we need not worry further about the assumption made.

Let class *I* impose a tax ξ per unit of goods produced. It receives altogether an amount ξu of money, which in terms of goods is equivalent to $\xi u/p$ units of goods. Class *II* produces u units of goods. It pays for those goods to class *III* the amount $q(u)$, which is equivalent to $q(u)/p$ units of goods. Moreover, it pays to class *I* an equivalent of $\xi u/p$ units of goods. It retains $u - q(u)/p - \xi u/p$ units. Class *III* gets an equivalent of $q(u)/p$ units of goods. The case, though at first sight very different from the one discussed before, can be thus expressed in terms similar to the former. We have

$$\frac{q(u)/p}{u} \backsim \theta, \quad \frac{\xi u/p}{u} \backsim \alpha. \tag{11}$$

The relations between θ and α assumed here are, however, different.

If class *II* adjusts the production so as to maximize its profits, we have (ref. 3, p. 50)

$$u = \frac{b' - (B + \xi)a'}{2 + 2Aa'}, \tag{12}$$

$$p=\frac{b'+2Aa'b'+(B+\xi)a'}{2a'(1+Aa')}. \tag{13}$$

Consider the case where class I determines the taxation in such a way as to make the amount $\xi u/p$ a maximum. Equations (12) and (13) give us the amount ξ_m of such a tax. The expression for ξ_m in that case can be obtained in closed form. However, it is rather clumsy and unusable. We may consider an alternate hypothesis, namely, that class I fixes ξ so as to maximize not the equivalent amount $\xi u/p$ of goods received, but the monetary rate ξu.

Thus we must consider in terms of money also the amounts retained by class II, namely, $pu - q(u) - \xi u$, and by class III, namely $q(u)$. This gives

$$\xi_m=\frac{b'-Ba'}{2a'}, \tag{14}$$

which is positive, since $b'-Ba'$ is greater than zero.[3] We also have

$$(\xi u)_{max}=\frac{(b'-Ba')^2}{8a'(1+a'A)}. \tag{15}$$

II

We may also consider the whole problem of interaction of classes I and II from a different angle.

Consider again two coexisting classes I and II. Let both classes, amongst other activities, perform two given activities A and B, producing correspondingly per unit time some results a and b. Those results may be considered as "commodities", either of a material nature, as food, or of a more abstract nature, as knowledge of some kind that may be communicated to others. Let the amounts of the results of activities produced by the first class be a_1 and b_1, the amounts produced by the second—a_2 and b_2. It may happen that a_1 is rather large while a_2 is small, and at the same time b_1 is small while b_2 is large. In that case an exchange of "commodities" will take place, the first class receiving some b from the second and giving in return some a.

To determine the character of that exchange, we shall use, as has been done by other authors[3,4], the concept of satisfaction, which we may apply to a class as a whole if the latter consists of approximately similar individuals. L. L. Thurstone[5] has shown that satisfaction, as a psychological quantity, may be actually, though indirectly, measured and discussed quantitatively. Thurstone comes to the conclusion, derived from psychological experimental evidence, that the

satisfaction varies logarithmically with the amount of commodity possessed, and he assumes moreover that for several commodities the satisfactions are simply additive. However, he makes some reservations as to the generality of the logarithmic relation. For the present general discussion, we shall not make any special assumptions about the shape of the satisfaction function. Moreover, we shall consider the satisfaction not in terms of the quantities of commodities possessed, but in terms of the rates of productions of those commodities. This is psychologically legitimate, for one may derive a greater satisfaction from producing or receiving per unit time more of a commodity. We shall therefore speak of "production" of a commodity by a class, even if that commodity is received from outside, for instance from the other class.

We assume that for any individual of class I there is a satisfaction function $s_1(x, y)$, where x and y are the amounts of the two commodities in question received per unit time. Similarly, for class II we have $s_2(x, y)$. If N_1 and N_2 are the numbers of individuals in classes I and II respectively, we shall define $S_1(x, y) = N_1 s_1(x, y)$ and $S_2(x, y) = N_2 s_2(x, y)$ as the satisfaction function for classes I and II respectively. We shall put

$$\frac{\partial S_1}{\partial x} = X_1(x, y); \quad \frac{\partial S_1}{\partial y} = Y_1(x, y); \tag{16}$$

$$\frac{\partial S_2}{\partial x} = X_2(x, y); \quad \frac{\partial S_2}{\partial y} = Y_2(x, y). \tag{17}$$

If each individual in a class, and therefore the class as a whole, agrees to such an exchange for which his satisfaction has the largest possible value, and if this exchange goes on in such a way that for a unit of a always the same number of units of b are given, then we may calculate the results of such an exchange by using formulae developed by G. Evans (see ref. 3, pages 125–128). Denoting by x_1 and y_1 the rates of production of x and y in class I when the exchange is operating, by x_2 and y_2 corresponding quantities for class II, and by a_1, b_1, a_2, and b_2 the corresponding quantities in the absence of exchange, we have for the determination of x_1, y_1, x_2 and y_2 the following equations:

$$\frac{X_1(x_1, y_1)}{X_2(a_1 + a_2 - x_1, b_1 + b_2 - y_1)} = \frac{Y_1(x_1, y_1)}{Y_2(a_1 + a_2 - x_1, b_1 + b_2 - y_1)} \tag{18}$$

$$\frac{y_1 - b_1}{x_1 - a_1} = -\frac{X_1(x_1, y_1)}{Y_1(x_1, y_1)}, \tag{19}$$

$$x_1 + x_2 = a_1 + a_2,$$

$$y_1 + y_2 = b_1 + b_2. \tag{20}$$

Equations (18), (19) and (20) express x_1, y_1, x_2 and y_2 in terms of a_1, b_1, a_2, and b_2;

$$x_1 = u_1(a_1, b_1, a_2, b_2); \quad x_2 = u_2(a_1, b_1, a_2, b_2);$$

$$y_1 = v_1(a_1, b_1, a_2, b_2); \quad y_2 = v_2(a_1, b_1, a_2, b_2). \tag{21}$$

Depending on the choice of the functions S_1 and S_2, a different distribution of rates of production will be obtained. It is possible that while $a_1 + b_1 > a_2 + b_2$, yet $x_1 + y_1 < x_2 + y_2$.

The quantities a_1, b_1, a_2 and b_2 refer to the class as a whole. The corresponding quantities per individual shall be denoted by a_1', a_2', b_1' and b_2'. Thus $a_1 = N_1 a_1'$, etc.

Let us consider that while class *II* supplies some material goods necessary for life, class *I* provides for goods of nonmaterial character which are, however, also necessary to the community. These may be the organization of military defense, legislation, policing, etc. Class *II* may agree to supply class *I* with a definite fraction of its products provided class *I* in its turn supplies it adequately with the useful results of its activities. The question may arise as to the units in which we shall measure the "goods" supplied by class *I*. While it may be difficult to give a general definition of such units for all specific instances, the problem is solved in daily life. A public officer is discharged when he does not perform his duties adequately. In other words, he does not receive in that case the remuneration which would be forthcoming to a competent person. In all such cases we have an exchange of nonmaterial goods against material. It is true that in some cases, as for instance in the case of a business executive, the performance of his duties may be directly measured by the number of dollars of profit which it brings to the business. In other cases, however, as for instance in the case of a judge or a policeman or an army officer, this is not possible. In every individual case, however, a definite criterion for measuring indirectly the dollar value of some activities does exist, and this defines a practical unit of such an activity. In general, we must consider that each class may produce both goods, but one class produces predominantly one type of goods, the other class—the other type.

Following L. L. Thurstone[4] we shall choose as an illustration for the satisfaction functions s_A and s_B the following expressions:

$$
\begin{aligned}
s_A &= A_1 \log \alpha_1 x + B_1 \log \beta_1 y , \\
s_B &= A_2 \log \alpha_2 x + B_2 \log \beta_2 y .
\end{aligned}
\tag{22}
$$

For the sake of simplicity we shall first put $A_1 = A_2 = B_1 = B_2 = \alpha_1 = \beta_1 = \alpha_2 = \beta_2 = 1$, so that $s_A = s_B = s$; in other words, we shall consider first the satisfaction function as being the same for all individuals and study only the effect of different ability to produce a given good or service.

We have

$$
S_1 = N_1 s , \quad S_2 = N_2 s , \tag{23}
$$

while X_1, Y_1, X_2, and Y_2 are defined by

$$
\frac{\partial S_1}{\partial x} = X_1 , \quad \frac{\partial S_1}{\partial y} = Y_1 , \quad \frac{\partial S_2}{\partial x} = X_2 , \quad \frac{\partial S_2}{\partial y} = Y_2 . \tag{24}
$$

Equations (18)-(20) now give:

$$
\begin{aligned}
x_1 &= \frac{2a_1 b_1 + a_1 b_2 + a_2 b_1}{2(b_1 + b_2)} ; \quad x_2 = \frac{2a_2 b_2 + a_1 b_2 + a_2 b_1}{2(b_1 + b_2)} ; \\
y_1 &= \frac{2a_1 b_1 + a_1 b_2 + a_2 b_1}{2(a_1 + a_2)} ; \quad y_2 = \frac{2a_2 b_2 + a_1 b_2 + a_2 b_1}{2(a_1 + a_2)} .
\end{aligned}
\tag{25}
$$

Introducing

$$
\frac{a_1}{a_2} = \lambda ; \quad \frac{b_1}{b_2} = \mu , \tag{26}
$$

we have

$$
\begin{aligned}
\frac{x_1}{x_2} &= \frac{2a_1 b_1 + a_1 b_2 + a_2 b_1}{2a_2 b_2 + a_1 b_2 + a_2 b_1} = \frac{y_1}{y_2} = \frac{2b_1 \lambda + b_2 \lambda + b_1}{2b_2 + b_2 \lambda + b_1} \\
&= \frac{2a_1 \mu + a_2 \mu + a_1}{2a_2 + a_2 \mu + a_1} .
\end{aligned}
\tag{27}
$$

We also have

$$
\begin{aligned}
\frac{\partial}{\partial \lambda} \left(\frac{x_1}{x_2} \right) &= \frac{2(b_1 + b_2)^2}{(2b_2 + b_1 + b_2 \lambda)^2} > 0 , \\
\frac{\partial}{\partial \mu} \left(\frac{x_1}{x_2} \right) &= \frac{2(a_1 + a_2)^2}{(2a_2 + a_1 + a_2 \mu)^2} > 0 .
\end{aligned}
\tag{28}
$$

The quantity x may represent the amount of legislative or military service produced by class I, while y represents the amount of material goods supplied by class II. Equations (25) give us these amounts in terms of a_1, a_2, b_1, b_2 and therefore in terms of N_1, N_2, a_1', a_2', b_1', and b_2'.

As a last illustration of the different types of interaction of social classes, we shall outline a theory of interaction between the industrial and agricultural population of a country, again, of course, under rather oversimplified assumptions.[2]

Let N_i denote the number of individuals working in industry, N_a —that in agriculture. Let the total population be

$$N = N_i + N_a . \tag{29}$$

Denote by p_i, p_a, respectively, the amount of goods produced per capita in industry and agriculture per unit time, expressed in some comparable units, as for instance in their dollar values. Both p_i and p_a will be functions of N_i and N_a, as well as of the supply of natural resources of the country. They can be divided very roughly into the total area S of land available and the resources R of ores, minerals, etc. Thus

$$p_i = f_i(N_i , N_a , R , S) ; \quad p_a = f_a(N_i , N_a , R , S). \tag{30}$$

Let c_i and c_a be the per capita consumptions in industry and agriculture, respectively, per unit time. Also let an exchange of goods between the two groups of individuals take place, so that the agricultural individuals supply the industrial with an amount $N_a G_a$ of goods, receiving in return a fraction θ of industrial goods. The quantities G_a and θ will be connected by some kind of demand equation, so that

$$G_a = f(\theta). \tag{31}$$

For the rate of change of the total wealth W_i and W_a of the industrial and agricultural population, we have, as in chapter v,

$$\begin{aligned} \frac{dW_i}{dt} &= N_i (p_i - c_i) + N_a G_a - \theta N_a G_a , \\ \frac{dW_a}{dt} &= N_a (p_a - c_a) - N_a G_a + \theta N_a G_a . \end{aligned} \tag{32}$$

The per capita rates w_i and w_a are

$$\begin{aligned} w_i &= p_i - c_i + \frac{N - N_i}{N_i} G_a (1 - \theta) , \\ w_a &= p_a - c_a - G_a (1 - \theta). \end{aligned} \tag{33}$$

If every individual chooses his occupation in industry or in agriculture according to whether w_i or w_a is larger, then there will be a shift of population to industrial occupations if $w_i > w_a$, and to agricultural if $w_i < w_a$. In equilibrium we have

$$w_i = w_a . \tag{34}$$

Because of expressions (29), (30) and (32), equation (34) gives a relation between N_i, N, R, S and θ. Or, introducing

$$\frac{N_i}{N} = \eta , \tag{35}$$

we may say that equation (34) furnishes a relation between N, η, R, S and θ. For fixed values of θ, N, R and S this relation gives us the value of η or, what is the same, the values of N_i and N_a. But, because of equations (33), this fixes the values of w_i and w_a as functions of θ. If every individual tries to make his $w_i = w_a$ a maximum, then $\partial w_i/\partial\theta = 0$ gives us an equation for the determination of θ. Once θ is determined as a function of η, N, R and S, we can express η as a function of N, R and S. Thus

$$\eta = \eta(N, R, S). \tag{36}$$

Because of expression (35), equations (30) may be written thus:

$$p_i = F_i(N, \eta, R, S); \quad p_a = F_a(N, \eta, R, S). \tag{37}$$

The total per capita rate of production of any goods is equal to

$$p.c. = \frac{N_i p_i + N_a p_a}{N} = \eta p_i + (1 - \eta) p_a , \tag{38}$$

and, because of equations (37), may be expressed through N, η, R, S. Data for N, η and S are readily available. The determination of R presents greater difficulties. In some cases we may assume very approximately $R \propto S$.

The foregoing constitutes quite a program for further mathematical research. To illustrate, however, what kind of relation may be thus obtained, let us consider the following crude example as an illustration. Equations (37) give us

$$\eta = \eta(N, R, S, p_i). \tag{39}$$

It is readily seen that in general η increases with p_i. Whatever the exact relation (39) between η and p_i, as a first approximation we may consider

$$\eta \propto p_i . \tag{40}$$

At the same time, for a given η, p_i will be the greater, the larger the natural resources available per capita. These will roughly vary inversely as the population density δ. Yet obviously even for δ tending to zero, p_i will not exceed a finite value, since it is limited by the ability of an individual to produce goods even with an infinite supply of raw material. We therefore put

$$p_i \propto \frac{\eta}{B+\delta}, \tag{41}$$

B being a constant.

If, as is usually the case, $p_a << p_i$, then, approximately, we have from equations (38):

$$p.c. = \eta p_i . \tag{42}$$

This, combined with expression (41), gives, with A as a coefficient:

$$p.c. = \frac{A\eta^2}{B+\delta}. \tag{43}$$

We may expect that $p.c.$ will vary approximately as the per capita income in the country. To what extent even the crude equation (43) is satisfied in some cases is shown in Figure 1, data for which have been

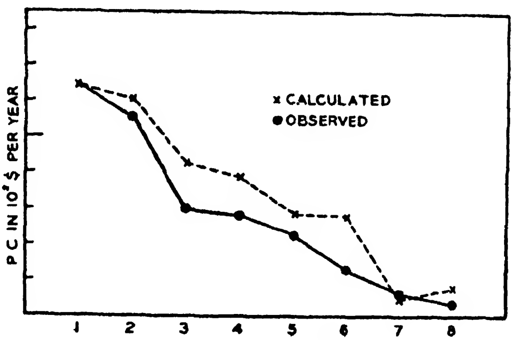

FIGURE 1

Values of $p.$ $c.$ for different countries 1—United States; 2—Canada; 3—Switzerland; 4—Norway; 5—Germany; 6—Finland; 7—Russia; 8—Japan. (Reproduced from *Psychometrika*, 1942, Volume 7, with the permission of the editors.)

compiled from different sources. That figure is by no means intended as corroborating equation (43), but rather to show how relations obtained by theoretical considerations may be compared with available data.

Countries having larger colonies are intentionally excluded from Figure 1. While data on η are available for most principal countries, none are available for colonies. There are also other complicating sit-

uations that enter into the case when we treat a problem of a country with a multiple-bounded contour. All this should stimulate the theoretical study of the general problem as outlined above. The exact expression of p_i in terms of η, δ and S, obtained from equations (40) and (41), will undoubtedly be much more complicated than equation (43), and it is that complicated equation which should be compared with actual data.

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CHAPTER VII

ANOTHER TYPE OF SOCIAL INTERACTION AND ITS POSSIBLE APPLICATIONS

Let us consider a case of interaction of two active groups of such a nature that each group opposes the behavior of the other the more strongly, the greater the success of that other group. This success may naturally be measured by the product of two factors: the ratio of the number of passive individuals who exhibit a given behavior to the total number of passives, and the average intensity of that behavior. Let the average intensity of behavior A be denoted by w_A. Then the total success of class A will be expressed by $\alpha_1 w_A\, x/N'$ where α_1 is a coefficient; or putting

$$\alpha_1 w_A = \varepsilon \,, \tag{1}$$

that success will be measured by $\varepsilon x/N'$. Similarly, if w_B denotes the average intensity of behavior B, then the success is given by $\alpha_2 w_B y/N'$ or, putting

$$\alpha_2 w_B = \varepsilon' \,, \tag{2}$$

by

$$\varepsilon' y/N' \,.$$

How to measure the quantities w_A and w_B is another question. We shall just consider the whole problem *in abstracto* and then give some possible concrete illustrations.

According to the above assumptions, using the same procedure as before, we put

$$\begin{aligned} a_0 &= a_0^* \left(1 + \varepsilon' \frac{y}{N'}\right) \,, \\ c_0 &= c_0^* \left(1 + \varepsilon \frac{x}{N'}\right) \,. \end{aligned} \tag{3}$$

The first expression (3) shows that the effort of class A increases as the success of class B increases. The second expression shows a corresponding relation for class B.

We now have, using a reasoning similar to that of chapter iii:

$$\frac{dx}{dt} = a_0^*\left(1 + \varepsilon' \frac{y}{N'}\right) x_0 + ax - c_0^*\left(1 + \varepsilon \frac{x}{N'}\right) y_0 - ay, \tag{4}$$

or, because of

$$x + y = N', \tag{5}$$

after rearrangements

$$\frac{dx}{dt} = \left(2a - a_0^*\varepsilon' \frac{x_0}{N'} - c_0^*\varepsilon \frac{y_0}{N'}\right) x + \left[\left(a_0^*\varepsilon' \frac{x_0}{N'} - a\right) N' + a_0^* x_0^* - c_0^* y_0^*\right]. \tag{6}$$

As t increases, the value of x tends asymptotically to

$$x = \frac{a_0^*\varepsilon' x_0 - aN' + a_0^* x_0 - c_0^* y_0}{a_0^*\varepsilon' \frac{x_0}{N'} + c_0^*\varepsilon \frac{y_0}{N'} - 2a}, \tag{7}$$

and is positive under the same assumptions as made in chapter iii. A similar expression is obtained for y and therefore

$$\frac{x}{y} = \frac{a_0^*\varepsilon' x_0 - aN' + a_0^* x_0 - c_0^* y_0}{c_0^*\varepsilon y_0 - aN' + c_0^* y_0 - a_0^* x_0}. \tag{8}$$

Equation (8) may be written

$$\frac{x}{y} = \frac{(a_0^*\varepsilon' + a_0^*)\dfrac{x_0}{y_0} - \left(c_0^* + \dfrac{aN'}{y_0}\right)}{\left(c_0^*\varepsilon + c_0^* - \dfrac{aN'}{y_0}\right) - a_0^* \dfrac{x_0}{y_0}}, \tag{9}$$

which is of the form

$$\frac{x}{y} = \frac{A \dfrac{x_0}{y_0} - B}{C - a_0^* \dfrac{x_0}{y_0}}. \tag{10}$$

Here $A > 0$ and $B > 0$. Since, with the above mentioned assumption, x and y are both non-negative, therefore $C > 0$. When $x_0/y_0 = C/a_0^*$, then $x/y = \infty$; in other words, $x = N'$, $y = 0$; and all of the passive population exhibit behavior A with an average intensity w_A. If y_0 is fixed, the requirement $x = N'$ gives

$$x_0 = Cy_0/a_0^* . \tag{11}$$

But c_0 is a linear function of $\varepsilon = \alpha_1 w_A$. Hence, the stronger the average intensity of activity A, the greater x_0 must be for a given y_0, in order to impress that activity on the whole passive population. If for a given x_0, w_A is too great, then the denominator of equation (10) will be positive and x/y will be finite; hence $x < N'$. If all the coefficients in our equations were known, then from a given maximum intensity w_A of behavior A which still can be imposed on *all* the passive individuals we could calculate x_0 for a known y_0.

We shall denote by w_{A_m} the maximum value of w_A which still can be impressed on the whole passive population. Correspondingly we shall put $\varepsilon_m = \alpha w_{A_m}$. For a given x_0 and y_0, ε_m is determined as the root of equation (11).

Let us now consider a simplified case, in which $w_B = 0$ and therefore $\varepsilon' = 0$. This means that group B merely resists the behavior A, but does not tend to impose any qualitatively different behavior B. In that case

$$\frac{x}{y} = \frac{a_0^* \dfrac{x_0}{y_0} - \left(c_0^* + \dfrac{aN'}{y_0}\right)}{(c_0^* \varepsilon + c_0^*) - \dfrac{aN'}{y_0} - a_0^* \dfrac{x_0}{y_0}} . \tag{12}$$

In order to have $x = N'$ or $x/y = \infty$, we must have

$$a_0^* \frac{x_0}{y_0} + \frac{aN'}{y_0} = c_0^*(1 + \varepsilon_m) . \tag{13}$$

Suppose now that the same group of x_0 active individuals tries to impose another behavior of intensity w_{A_1} on the passive population. This time let behavior A_1 be opposed by a different group of $y_{01} = \lambda y_0$ other active individuals, the coefficients of influence remaining the same. Then, denoting by x_1 and y_1 the number of passive individuals that correspondingly exhibit and do not exhibit behavior A_1, we have an expression similar to equation (12), in which ε_1 is put instead of ε and λy_0 instead of y_0. Introducing equation (13) into that expression we find after elementary calculations:

$$\frac{y_1}{x_1 + y_1} = \frac{\lambda c_0^*(1 + \varepsilon_1) - c_0^*(1 - \varepsilon_m)}{\lambda c_0^*(1 + \varepsilon_1) - \left(\lambda c_0^* + \dfrac{2aN'}{y_0}\right)} , \tag{14}$$

which is of the form

$$\frac{y_1}{x_1 + y_1} = \frac{A' - B'\varepsilon_m}{C'}. \tag{15}$$

Let us discuss a case considered in chapter iii, namely, that of two active classes I and II, and a passive class III. Again let class I represent the "controlling" or "governing class" and let x_0 refer to it. Class II again is the one which organizes the production of goods. We discussed in chapter vi some cases of interaction of two such classes, where class I requires a certain amount of goods produced by classes II and III to be surrendered to it. Here we shall consider the situation from a somewhat different point of view. Class I may require that every individual of class II and III give a certain fraction of everything he produces to class I. Class II will oppose it, the opposition being the stronger, the greater the fraction required. Thus that fraction may be used as a measure of w_A. Class I will impose as high a w_{A_m} as can be impressed on the whole population, and that fraction will be the larger, the larger x_0. In practice we may take as an illustration for w_{A_m} the ratios of the governmental tax receipts to the total national income.

If we consider the case of very small a, so that the latter can be neglected, then equations (9), (13) and (14) become simplified. In particular, because of expression (1), equation (13) now becomes of the form

$$\frac{x_0}{y_0} = A'' + B''w_{A_m}, \tag{16}$$

while equation (15) becomes

$$\frac{y_1}{x_1 + y_1} = \frac{A''' - B'''w_{A_m}}{C'''}. \tag{17}$$

Ascribing to w_{A_m} the above meaning, we may try to check equation (16), if we have some other means of determining x_0/y_0. The following gives a very rough possible estimate of that ratio. Most of the active population of a country is concentrated in cities. The stronger the "governing" class I, the more centralized the government and the larger the relative size of the capital city. Denoting by N_c the population of the capital, and by N_u the total urban population, we may consider approximately

$$\frac{x_0}{y_0} \backsim \frac{N_c}{N_u - N_c}. \tag{18}$$

Assuming that for different countries the coefficients A'' and B'' are the same, which is of course only an extremely rough approximation, we shall expect

$$\frac{N_c}{N_u - N_c} \sim A'' + B'' w_{A_m}. \tag{19}$$

Data of w_{A_m} are scarce and inaccurate[2]. Using what is available, the result of comparison of equation (19) with observation is shown in Figure (1), with data valid about 1930.

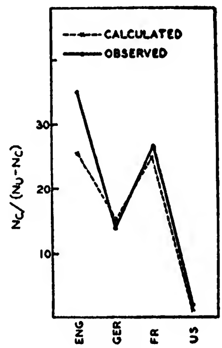

FIGURE 1

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It must also be kept in mind that the whole discussion is based upon considerations of steady equilibrium states. A sudden increase of w_{A_m} will not result in an immediate variation of $N_c/(N_u - N_c)$ according to equation (19). There will be considerable time lags governed by the general differential equation (4), in which w_A and hence ε are made explicit functions of time.

Equation (17) may be used for estimating the success of imposing other behaviors by x_0 in terms of w_{A_m}. The governing class requires certain standards of behavior in different lines of life, the requirements being put into effect with different amounts of effort for different types of behavior. Deviation from the required behavior constitutes crimes of various degrees. Thus equation (17) may be used to study comparative criminality in different populations, for $y_1/(x_1 + y_1)$ denotes the ratio of individuals who *do not* obey the dictates of class I. The quantity y_1 here denotes the total number of criminals, while y_{01} denotes the number of criminals belonging to the active class. It will be agreed that a large number of crimes are made by passive individuals as a result of imitation, etc.

Most criminal statistics are not very accurate, and are hardly comparable for different countries due to different legal standards.[2] Incidents of such obviously criminal acts as murders would be perhaps the most comparable. An important factor, however, has to be added in that case to equation (17). Other conditions, including x_0/y_{01}, being the same, there will be a difference in the incidence of crime depending on the ease with which a crime is, or can be, hidden. Denote the probability of a successful concealment of a crime by f. The latter depends on the density of population δ. This function $f(\delta)$ of δ is a rather complicated one. Obviously $f(\delta)$ must be zero for $\delta = 0$ as no crimes are committed in an unpopulated country. Yet beginning with rather small values of δ, $f(\delta)$ must, within a certain range of δ's, decrease with δ, approximately as $1/\delta$; for the greater the density of population, the larger number of individuals a crime affects, and the sooner it becomes known. A murder of an individual living alone in a secluded spot may remain undiscovered for weeks. As δ increases further, however, this also increases the ease with which the criminal can hide himself after the deed. Thus $f(\delta)$ must start at zero, reach a maximum, decrease approximately as $1/\delta$, and then increase again. This latter increase is likely a factor of increased crime incidence in large cities, although a concentration of the active elements in cities, as mentioned above, undoubtedly plays an important part too. The density of population in large cities is of the order of 10^4 individuals per square kilometer or higher. The highest population densities in countries as a whole are about 300–400 ind. per square kilometer. Thus it is plausible to assume that the increase of $f(\delta)$ begins only above values of δ of the order of 10^3 individuals per square kilometer. The incidence of crime CR will be proportional to $y_1/(x_1 + y_1)$ and to $1/\delta$. Using for $y_1/(x_1 + y_1)$ expression (17), we find

$$CR = \frac{D - w_{A_m}}{\delta}, \tag{20}$$

where D is a constant. The comparison of this expression with observation is shown in Figure (2).

We may consider a still different type of behavior, which is not forbidden by class I, but is not too much encouraged. As an example we may cite divorce. If $y_1/(x_1 + y_1)$ is very small, in other words if $y_1 << x_1$, we may substitute for $y_1/(x_1 + y_1)$, simply y_1/x_1. For that ratio we have an expression of the same form as (10), namely

$$\frac{y_1}{x_1}=\frac{\bar{A}\dfrac{y_{01}}{x_0}-\bar{B}}{\bar{C}-\dfrac{y_{01}}{x_0}}, \tag{21}$$

where y_{01} is the number of active individuals advocating divorce. As before, we have $y_{01}=\lambda y_0$. When y_1/x_1 is small, then we may expand the right side of equation (21) and stop at the linear term. Thus we obtain

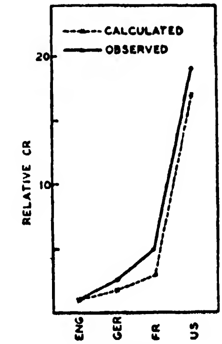

FIGURE 2

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$$DR=\frac{y_1}{x_1}=\bar{A}\frac{y_0}{x_0}-\bar{B}'. \tag{22}$$

Combining equation (22) with (18) we find a relation between y_1/x_1 and $(N_u-N_c)/N_c$, which is illustrated in Figure (3)*.

It must be emphasized that the above illustrations do not mean any "confirmation" of a particular theory. They merely serve to illustrate how, starting from purely theoretical abstract concepts, we may gradually arrive at relations that can be tested by observation. To

* In the illustration in Figure 3 to equation (19), w_{A_m} for the United States was taken as the ratio of the total federal tax receipts to the national income. Accordingly, the population of Washington, D.C. was taken for N_c. Considering that due to the decentralized system of the U. S. Government such regulations as concern divorce are more of a state nature, in computing data in Figure 3 by means of equations (22) and (18) N_c is taken as representing the sum of populations of all state capitals.

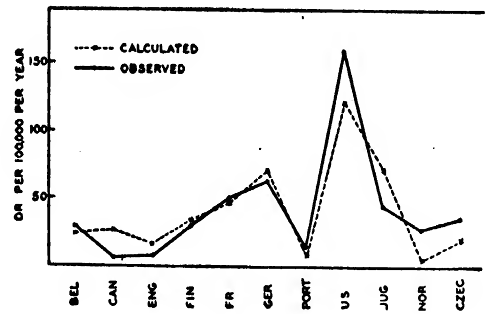

FIGURE 3

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speak of actual verifications of any such theory would require much more elaboration of the theory, which must take into account many more complex factors. The above illustrations show, however, how certain relations may be suggested even by an inadequate theory, which thus helps us to notice such relations, for we usually notice only what we look for, and we look for things which we expect.

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CHAPTER VIII

PERIODIC FLUCTUATIONS IN THE BEHAVIOR OF A SOCIAL GROUP

Let us consider here the case in which the increase of the effort is determined not by the instantaneous value of the attained success, but by the integral success over all of the past time. In other words, the effort of the active individual is affected by the total success in the past.

The more remote the success in the past, the less we would expect it to affect the effort of the individual. Using the same notations and denoting by $f_1(t-\tau)$ and $f_2(t-\tau)$ two functions of the present time t and a past time τ, of such a nature that $f_1(t-\tau)$ and $f_2(t-\tau)$ decrease with increasing $t-\tau$, we may express now the coefficients of influence a_0 and c_0 in the form:

$$\begin{aligned} a_0(t) &= a_0^*[1-\varepsilon\int_{-\infty}^{t} x(\tau)\, f_1(t-\tau)\, d\tau]; \\ c_0(t) &= c_0^*[1-\varepsilon'\int_{-\infty}^{t} y(\tau)\, f_2(t-\tau)\, d\tau]. \end{aligned} \tag{1}$$

Instead of equation (2) of chapter iii, we now have

$$\begin{aligned} \frac{dx}{dt} = a_0^*[1-\varepsilon\int_{-\infty}^{t} x(\tau)\, f_1(t-\tau)\, d\tau]x_0 + ax - c_0^*\,[1 - \\ \varepsilon'\int_{-\infty}^{t} y(\tau)\, f_2(t-\tau)\, d\tau]\, y_0 - ay, \end{aligned} \tag{2}$$

or, because of

$$x + y = N', \tag{3}$$

$$\begin{aligned} \frac{dx}{dt} = a_0^*\,[1-\varepsilon\int_{-\infty}^{t} x(\tau)\, f_1(t-\tau)\, d\tau]\, x_0 + 2ax - \\ c_0^*\,\{1-\varepsilon'\int_{-\infty}^{t} [N' - x(\tau)]\, f_2(t-\tau)\, d\tau\}\, y_0 - aN'. \end{aligned} \tag{4}$$

As an illustration, we shall solve equation (4) for a particularly simple case, namely

$$f_1(t-\tau) = f_2(t-\alpha) = e^{-a(t-\tau)}, \tag{5}$$

where a is a constant.

Equation (4) may be written:

$$\begin{gathered}\frac{dx}{dt} = a_0{}^* x_0 - a_0{}^* x_0\, \varepsilon\, e^{-at} \int_{-\infty}^{t} x(\tau)\, e^{a\tau}\, d\tau - c_0{}^* y_0 + \\ c_0{}^* y_0\, \varepsilon'\, e^{-at} \int_{-\infty}^{t} [N' - x(\tau)]\, e^{a\tau}\, d\tau + 2ax - aN' .\end{gathered} \tag{6}$$

Multiplying first by e^{at}, then differentiating with respect to t, remembering that

$$\frac{d}{dt} \int_{-\infty}^{t} x(\tau)\, e^{a\tau}\, d\tau = x(t)\, e^{at},$$

then shortening everything by e^{at} and rearranging, we find

$$\begin{gathered}\frac{d^2x}{dt^2} + (a - 2a)\frac{dx}{dt} + (a_0{}^* x_0\, \varepsilon + c_0{}^* y_0\, \varepsilon' - 2aa)\, x - \\ (aa_0{}^* x_0 - ac_0{}^* y_0 + c_0{}^* y_0\, \varepsilon' N' - aaN') = 0 .\end{gathered} \tag{7}$$

Let

$$\begin{gathered}a_0{}^* x_0\, \varepsilon + c_0{}^* y_0\, \varepsilon' - 2aa = A , \\ aa_0{}^* x_0 - ac_0{}^* y_0 + c_0{}^* y_0\, \varepsilon' N' - aaN' = B .\end{gathered} \tag{8}$$

If

$$0 < (a - 2a)^2 < 4A , \tag{9}$$

then equation (7) represents a damped oscillation around a value $x = \bar{x}$, of the form:

$$x = \bar{x} + c_1\, e^{-(a-2a)t} \sin(\nu t + \delta), \tag{10}$$

where

$$\nu = \sqrt{(a - 2a)^2 - 4A}, \tag{11}$$

while c_1 and δ are determined by the initial conditions. The value $\bar{x}$ of x, around which it oscillates and to which it tends as the amplitude decreases, is equal to

$$\bar{x} = \frac{B}{A} = \frac{aa_0{}^* x_0 - ac_0{}^* y_0 + c_0{}^* y_0\, \varepsilon' N' - aaN'}{a_0{}^* x_0\, \varepsilon_0 + c_0{}^* y_0\, \varepsilon' - 2aa}. \tag{12}$$

If inequalities (9) are not satisfied, the value (12) is approached aperiodically.

Thus for constant values of x_0 and y_0, the values of x and y, and hence of x/y, will fluctuate periodically. While for some values of the constants x may always remain less than y, for some other values x may periodically exceed y and again drop below it. Thus the type of behavior characteristic of the "majority" of the passive population will periodically fluctuate.

When α tends to infinity, then $e^{-\alpha(t-\tau)}$ is large only for values of τ in the neighborhood of t. But the value of

$$\int_{-\infty}^{t} x(\tau)\, e^{-\alpha(t-\tau)}\, d\tau$$

tends to zero. The case reduces to that of equation (6) of chapter iii. Indeed, making $\alpha = \infty$, we obtain from (12):

$$x = \frac{aN' + c_0^{*} y_0 - a_0^{*} x_0}{2a}. \tag{13}$$

The numerator of equation (13) is identical with the expression in parentheses in equation (6) of chapter iii. When this expression is positive, equation (6) of chapter iii has an equilibrium configuration given by equation (13). If it is negative, then equation (13) means physically that $x = 0$, $y = N'$, and we have again the case discussed on page 28.

When $\alpha = 2a$, the oscillations are undamped. For $\alpha < 2a$ we have a negative damping, the amplitude increasing indefinitely. Since both x and y are finite and cannot exceed N', the physical interpretation of this case would be, for instance, that x increases in an oscillatory way until it becomes equal to N'. If x reaches N' just at the moment of the maximum, when $dx/dt = 0$, then it will go down to zero and reach zero when $dx/dt < 0$. In other words, y will reach N' when $dy/dt > 0$. Due to the symmetry of x and y, we may therefore confine ourselves to the consideration of the case where x attains the value N' when $dx/dt > 0$. Let that happen at $t = t_1$. From then on x must remain constant, at least for a while, since it cannot increase and it cannot begin to decrease immediately because at that time the right side of equation (6) is positive. As x remains constant and equal to N', the second term of the right side of equation (6) will increase, tending for large values of t to $N'\, a_0^{*}\, x_0\, \varepsilon/\alpha$, while the fourth term tends to zero.

If α is sufficiently small, then after a sufficient time has passed after t_1, so that the second term is near enough to $N' a_0^* x_0 \varepsilon/\alpha$ and the fourth to zero, we shall necessarily have

$$a_0^* x_0 - c_0^* y_0 + aN' - \frac{N' a_0^* x_0 \varepsilon}{\alpha} < 0. \qquad (14)$$

Hence the right side of equation (6) will become negative and x will begin to decrease. If

$$\bar{x} \backsim \frac{N'}{2}, \qquad (15)$$

then x will swing into the other extreme of $x = 0$, $y = N'$. The same argument applies to that situation, except that the roles of the second and fourth terms are interchanged. If α is so small that

$$a_0^* x_0 - c_0^* y_0 - aN' + \frac{c_0^* y_0 \varepsilon' N'}{\alpha} > 0, \qquad (16)$$

then after remaining for a while equal to zero, x will increase again to the value N', provided $\bar{x}$ is in the neighborhood of $N'/2$. Thus if α is so small that both inequalities (14) and (16) hold, the behavior of the passive individuals will be swinging between behavior A and B, back and forth. Except during the transition periods, all individuals will exhibit behavior A for a while, then all swing for a while to a behavior B, and so forth.

If relations (14) and (15) are satisfied but (16) is not, then after the value $x = 0$ has been reached it will persist. All individauls will exhibit behavior B. On the other hand, if relation (15) is satisfied but (14) is not, x will remain equal to N' once it reaches this value. All passive individuals will exhibit behavior A.

If $\bar{x}$ is much larger than $N'/2$, then x, decreasing from $x = N'$, may not reach $x = 0$ before it begins to increase again. The swinging will go on between $x = N'$ and some finite value $x = x_1$. *Mutatis mutandis,* this holds for the reverse case when $\bar{x}$ is much smaller than $N/2$ and when $x = 0$ is reached first.

Oscillations of group behavior from one extreme to another are sometimes actually observed. It must be emphasized, however, that the assumption (5) is rather artificial and that until more complex cases have been investigated, no practical applications can be made.

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CHAPTER IX

SUGGESTIONS FOR A MATHEMATICAL THEORY OF INDIVIDUAL FREEDOM

In chapter vi a possible correlation between the activities of a typically organizing class and a general tolerance to behavior of others was suggested. It was pointed out that the imposition of the activity of the organizing class upon the passive population is based primarily on "free mutual agreement" (page 53). This raises the question of the general conception of freedom, which is frequently discussed in sociological literature.

Quantitative definitions of freedom have been attempted before. P. Sorokin[1] defines it as the ratio of the sum total of the means to satisfy our desires to the sum total of the desires. He does not, however, make any quantitative applications of this definition. It seems to be more advantageous to use a less general definition of freedom, defining it differently in its different aspects. In this chapter we shall consider two such aspects, without implying that these are the only two possible ones.[2]

I

We may first consider "freedom" from the point of view which perhaps is best termed "economic." Suppose an individual can perform physically a maximum amount w_0 of work per unit time. In general, he will perform on the average a lesser amount of work w per unit time, w being determined by his requirements for the necessities of daily life. We may then define the "economic" freedom F_e of an individual by the expression

$$F_e = \frac{w_0 - w}{w_0}. \tag{1}$$

The less work a person has to perform to keep himself alive, the freer he is. According to our definition, his freedom is zero when the amount of work which he has to perform is the maximum physically possible.

The necessary work w is in general determined by the social structure of the group of which the individual is a part, and there-

fore his freedom is also determined by that social structure.

Let us, for instance, consider the situation discussed in chapter v in which one class (I) organizes and directs the work of another class (II). The condition for the possibility of existence of class II is, using the same notations as in chapter v and denoting by S_2 the amount of goods accumulated per unit time per individual of class II, (equation 11, chapter v):

$$S_2 = \varepsilon_2 + \theta w \geqq 0 . \tag{2}$$

If this inequality does not hold, then, according to equation (11) of chapter v, dW_2/dt will be negative and class II could not exist permanently. From equation (2) we have

$$w = \frac{S_2 - \varepsilon_2}{\theta}. \tag{3}$$

Introducing equation (3) into (1) we find:

$$F_e = 1 - \frac{S_2 - \varepsilon_2}{w_0 \theta}. \tag{4}$$

For a given "price" θ, the economic freedom of an individual of class II decreases with increasing amount of goods, S_2, which he accumulates per unit time, and decreases also with the amount ε_2, which measures the ability of the individual of class II to produce goods without the organizing direction of the individuals of class I. The less an individual of class II is capable of producing himself, without the organizing direction of others, the less free he is.

If we suppose that the individuals of class I fix the value θ so as to make their gain a maximum, then θ is not arbitrary but is given by equation (16) of chapter v, namely:

$$\theta = \sqrt{\frac{bf(\eta)}{w_0}}, \tag{5}$$

where b is a constant, and $f(\eta)$ a function of the ratio of population of the two classes, specified more exactly in chapter v. Furthermore, in this case S_2 is nothing else than the expression in brackets in equation (18), chapter v; hence

$$S_2 = \varepsilon_2 = -b + \sqrt{bw_0 f(\eta)}. \tag{6}$$

Introducing expressions (5) and (6) into (4), we find:

$$F_e = \sqrt{\frac{b}{w_0 f(\eta)}}. \tag{7}$$

Equation (7) at first sight seems to contradict equation (1), for according to the latter, F_e increases with w_0, tending to 1 for $w_0 = \infty$. This paradox is resolved by the remark that equation (7) is obtained from the more general equation (1) by making specific assumptions about w. Equation (7) is derived by using equation (14) of chapter v. But according to that equation, w is itself a function of w_0.

We may now study, for instance, the dependence of F_e on various parameters, such as the total population, etc. The quantity $f(\eta)$ measures essentially the amount of goods produced for constant amount of labor expended. But this amount of goods will depend on other things, too; for instance, on the amount of raw material present per person, which will be inversely proportional to $N = N_1 + N_2$. Thus $f(\eta) \propto 1/N$. Similarly, the constant b will depend in some way on N, so that $b = b(N)$. The study of the personal economic freedom in terms of population density can thus be made. However, it will first involve the study of the dependence of $f(\eta)$ and of b on that density.

If, instead of the above situation, we consider the other situation discussed in chapter v, we would obtain a different expression for F_e. In all of the three situations discussed in chapter v, we could also express the economic freedom F_{e_1} of individuals of class I, as well as the freedom F_{e_2} of the individuals of class II, for in all these cases we can express the amount of work w_1 and w_2 given by individuals of classes I and II, respectively, in terms of the corresponding w_{01} and w_{02}. Thus we have a way of expressing the individual "economic" freedom for the different social situations assumed.

II

We shall now consider a different aspect of individual freedom, not involving economic relations directly.

Let any person in a social group have the possibility of performing either one or several of the n different activities $A_1, A_2, \cdots A_n$. Furthermore, let a given individual like some m activities out of those n and dislike the other $n - m$. There are altogether

$$M = \sum_{m=0}^{m=n} \frac{n!}{m!\,(n-m)!} \tag{8}$$

ways in which different pleasant activities may be chosen by the individual for there are $\dfrac{n!}{m!\,(n-m)!}$ ways of choosing m activities from n, and m itself may vary from 0 to n. If the "liking" of dif-

ferent choices is distributed at random, then altogether the fraction $1/M$ of all the individuals will like a particular choice of m activities and dislike all others.

Consider, in a population of N individuals, one who has *that particular choice* of activities. That individual comes into social contact with a certain number of his fellow individuals per unit time. The frequency with which he comes into such a social contact is proportional to the total number N of individuals per unit area, and may be expressed as

$$aN\,, \tag{9}$$

where a is a constant, depending on different external conditions such as ways of communication, etc. Let τ denote the average fraction of a unit time, let us say, of a day, which an individual spends in contact with others. Then the fraction of time which a person spends in contact with others is

$$t_c = a\tau N\,, \tag{10}$$

while the fraction of time which he has entirely to himself, and which may be called "free," is given by

$$t_f = 1 - a\tau N\,. \tag{11}$$

During the fraction of time t_f the individual may indulge entirely in the activity of his own choice, since such an activity does not interfere with that of anyone else.

On the other hand, during the fraction of time t_c he can indulge unrestrictedly in the activity of his own choice only when he meets individuals who enjoy the same activities. For, while he is in contact with other individuals who have different tastes, some of his m activities may interfere with theirs and he must therefore restrict himself in that respect. But, as we have seen, altogether $1/M$ individuals choose the same activities. If we again assume a random distribution of the preferences amongst the individuals with whom the given individual comes into contact, we find that from the fraction t_c of the time he spends with them, a fraction t_c/M may be spent in indulging in the activities of his own choice. We may now define as the freedom F of the individual that fraction of his total time during which he is free to do what he wants to do. We then find

$$F = t_f + \frac{t_c}{M} = 1 - \frac{M-1}{M}\, a\tau N\,. \tag{12}$$

The freedom of an individual thus decreases with increasing N.

We may consider a more complex case by introducing intermediate situations. Suppose a given individual, who chooses in particu-

lar activities $A_1, A_2, \cdots, A_{m-1}, A_m$, is in contact with another individual, who chooses another set of m activities, $A'_1, A'_2, \cdots A'_m$. Let m'' of those activities be common to both. We may then say that the first individual has to restrict himself to the amount m''/m, being free to the amount $(m - m'')/m$. In that case, in order to obtain the expression for F, we should add to expression (12) another term of a rather complex structure, obtained by summation of all values $(m'' - m)/m$ taken over all possible combinations of choices which have common elements with a given one. In this way F also becomes a function of the particular choice of activities which an individual makes, and therefore F will vary from individual to individual.

Still more complex situations may be studied if we consider some distribution function for the preferred choices, so that certain preferences occur more frequently than others. This leads to rather interesting mathematical problems. Further studies will open many interesting possibilities.

The illustrations above show that even such a "purely sociological" concept as that of individual freedom may be made the subject of an exact mathematical treatment.

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CHAPTER X

SPATIAL DISTRIBUTION OF INDIVIDUALS IN A SOCIETY

We have considered in previous chapters various types of social interaction in which a population of individuals has been divided into two or more classes. However, hitherto, we have not paid any attention to the spatial distribution of the individuals belonging to different classes. In this chapter we shall illustrate by a few simple examples how possible effects of different spatial distribution could be studied mathematically. Again we select intentionally oversimplified examples, which in their simple form may not always have a counterpart in reality. We are here concerned not so much with deriving equations for actual cases as with illustrating the methodological principles.

We shall discuss here a case which may in its eventual development throw some light on the relation between rural and urban populations.

Suppose that out of N individuals a number N_r draw their means of existence from cultivation of land. Let them be distributed on the average uniformly through an area suitable for producing the needed amount of land products. On the other hand, let the N_u remaining individuals gather together in cities and produce goods that do not require *directly* any land for their production. We have

$$N = N_u + N_r . \tag{1}$$

Let the amount of goods produced by a person on the land be p_r per unit time, while the corresponding amount produced by a person in the city be p_u. In general p_r and p_u are not constant, but both may be functions of N_r and N_u, thus

$$p_u = f_u(N_u , N_r) ; \quad p_r = f_r(N_u , N_r). \tag{2}$$

Thus, for instance, for a land very scarcely populated and not used to the full extent, p_r may be approximately constant and determined by a constant fertility of the soil and by the average constant capacity of a man to produce work. However, as N_r increases, the total amount $N_r p_r$ of goods produced must necessarily tend to a constant value, determined by the *maximum possible* fertility of the soil. Hence p_r must

decrease with increasing N_r. The law of decrease will of course be rather complex and depend on many factors. As an illustration, we may consider just a simple form

$$p_r = \frac{a_1}{N_r + a_2}, \tag{3}$$

where a_1 and a_2 are constants. In general, however, the value of p_r may depend also on N_u.

Now we may consider different cases of behavior of the individuals. A simple case is when every individual tries to produce as much of any goods as he can. If for a given N_r and $N_u = N - N_r$, $p_u > p_r$, then individuals from the country will migrate to the city until, due to the change of N_u and N_r produced by this migration, p_u becomes equal to p_r. If $p_u < p_r$, a migration from the city to the country will take place. Since, because of equation (1), $N_r = N - N_u$, p_u and p_r in expressions (2) are functions of N and N_u. Hence the requirement

$$p_u = p_r \tag{4}$$

gives us an equation for the determination of N_u for a given total N.

To illustrate, let us assume that in the city p_u = Const., while p_r is given by (3). Since the maximum value of p_r is equal to a_1/a_2, and p_r decreases monotonically with increasing N_r, in order for equation (4) to be possible at all, we must have

$$\frac{a_1}{a_2} > p_u, \tag{5}$$

or

$$a_2 - \frac{a_1}{p_u} < 0. \tag{6}$$

Put

$$a_2 - \frac{a_1}{p_u} = -c^2. \tag{7}$$

Equations (4) and (3) give:

$$N_r = \frac{a_1}{p_u} - a_2, \tag{8}$$

or because of (1):

$$N_u = N + a_2 - \frac{a_1}{p_u}. \tag{9}$$

From equations (9) and (7) we have:

$$\frac{N_u}{N} = 1 - \frac{c^2}{N}. \tag{10}$$

Thus, the ratio of urban population to the total population increases with increasing total population N, tending to 1 for $N = \infty$.

We may consider a different mode of behavior of the individuals. For instance, as a purely theoretical example, we may consider the case where every individual tries to increase the total amount

$$G = N_u p_u + N_r p_r \tag{11}$$

of goods produced, and according to this aim chooses either the urban or the rural occupation. In this case we have to look for such a value of N_u which makes G a maximum. This is given by the equation

$$\frac{dG}{dN_u} = 0, \tag{12}$$

into which we substitute for p_u and p_r the expressions (2), after expressing N_r in terms of N and N_u by means of equation (1).

Again using for p_r the expression (3) and putting $p_u =$ Const., equation (12) becomes:

$$\frac{d}{dN_u}\left(N_u p_u + \frac{a_1(N - N_u)}{N - N_u + a_2}\right) = 0, \tag{13}$$

which gives, after elementary calculations,

$$N_u = N + a_2 - \sqrt{\frac{a_1 a_2}{p_u}}. \tag{14}$$

From equation (14) we have

$$\frac{N_u}{N} = 1 + \frac{1}{N}\left(a_2 - \sqrt{\frac{a_1 a_2}{p_u}}\right). \tag{15}$$

Because of inequality (6) the expression in parentheses is negative. For

$$N > \sqrt{\frac{a_1 a_2}{p_u}} - a_2 \tag{16}$$

the ratio N_u/N is positive and increases with N, becoming equal to 1 for $N = \infty$. The reversal of the inequality sign in (16) gives $N_u/N < 0$, which is physically impossible. In this case, N_u actually will be equal to zero, which means that all population in that case is rural.

A slightly different relation between N_u and N is obtained by

generalizing somewhat our assumptions. Instead of putting $p_u =$ Const., we may put:

$$p_u = \frac{b_1}{N_u + b_2}, \quad b_1 > 0, \quad b_2 > 0. \tag{17}$$

Using the first criterion and therefore setting $p_r = p_u$, we find in a similar way as before

$$\frac{N_u}{N} = A - \frac{B}{N}, \tag{18}$$

with

$$A = \frac{b_1}{a_1 + b_1}, \quad B = \frac{a_1 b_2 - a_2 b_1}{a_1 + b_1}. \tag{19}$$

While according to equation (10) N_u/N tends with increasing N to unity, according to equation (18) it tends to $A < 1$.

Figure 1 shows the comparison of equation (18) with data for

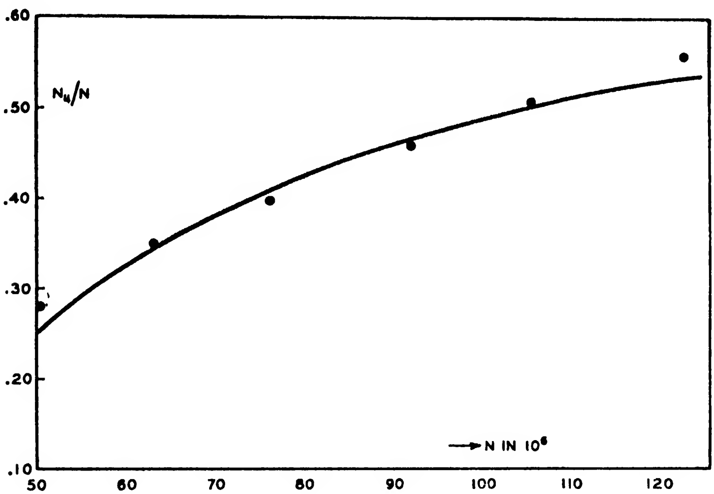

FIGURE 1

The curve is the graph of equation (18) with $A = .722$; $B = 23.7 \times 10^6$. The points represent observed data for the United States. (Reproduced from *Psychometrika*, 1942, Volume 7, with the permission of the editors.)

the United States (Reference 1, page 227). For Germany the simpler equation (10) represents the facts well (Figure 2) (Reference 2, page 14). The latter requires that $N_r =$ Const., all increase in population going into cities. This is the case with Germany, since between 1871 and 1933, the rural population remained practically constant, fluctuating between the extremes of $21{,}623 \times 10^3$ and $22{,}709 \times 10^3$. It

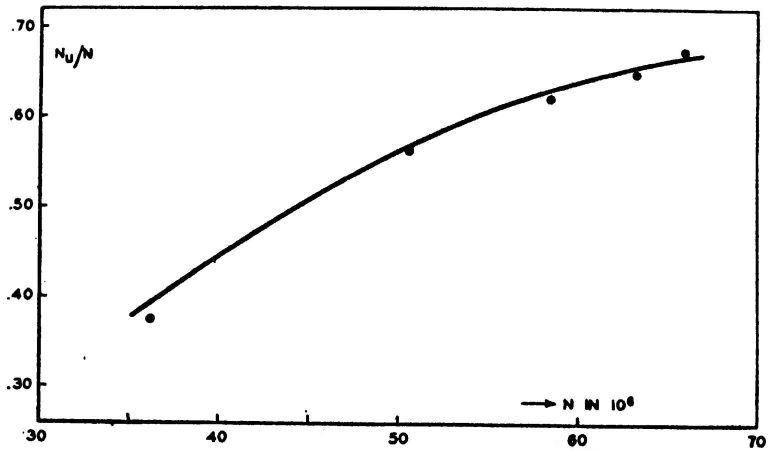

FIGURE 2

The curve is the graph of equation (10) with $c^2 = 22 \times 10^6$. The points represent observations for Germany. (Reproduced from *Psychometrika*, 1942, Volume 7, with the permission of the editors.)

may be asked why available data for the ratio of N_u/N in the United States for a period from 1800 were not used. The reason is this: the constants a_1 and a_2 are by their definition functions of the total area of the country, and the latter gradually increased in the United States during the first three quarters of the last century. A more detailed theoretical study is required, to be applicable to such a case.

Figure 3 represents data for Russia.[3] The actual data for Russia

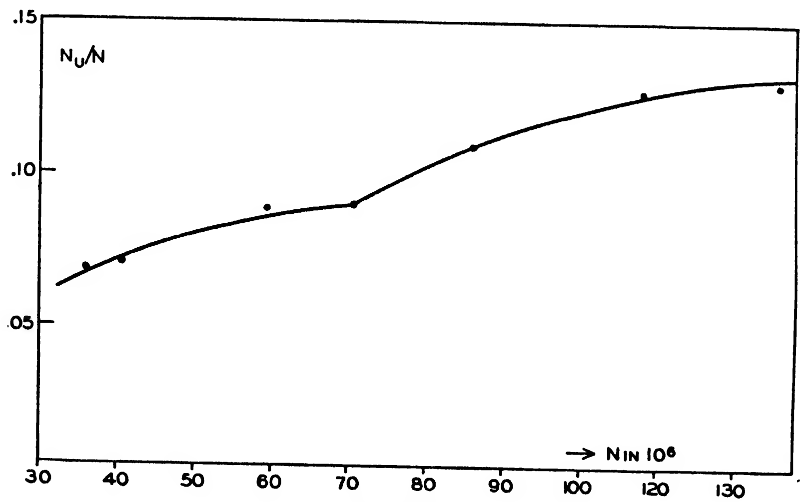

FIGURE 3

Comparison of theory and observation for Russia. Explanation in text.

may be represented by two separate curves of the type given by equation (18). The lower branch is represented by taking $A = \cdot 114$, $B = 1{\cdot}68 \times 10^6$; the upper, by taking $A = \cdot 18$, $B = 6 \times 10^6$. The point of discontinuity falls approximately at $N = 67 \times 10^6$. This value of the total population occurred in Russia about 1860. In 1862 serfdom was abolished in Russia, a reform which would have as a natural consequence an increase of rural-urban displacement, as well as a thorough change of many economic conditions. Hence the sudden change in the constants A and B at this point should be expected.

Data for Sweden[1] show a rather peculiar behavior (Figure 4),

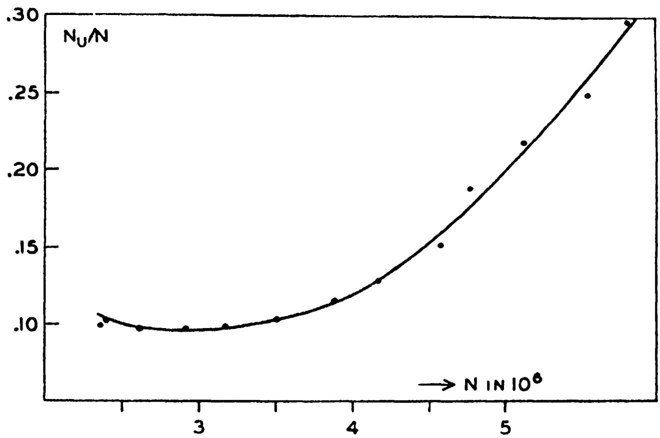

FIGURE 4
Data for Sweden. Explanation in text.

the ratio N_u/N apparently slightly decreasing at first, then increasing. This shows that other more complex factors, which have been left out of consideration here, have played a role in that country.

We may consider a somewhat more complicated situation. Let each urban individual produce p_u and consume c_u per unit time, and each rural individual produce p_r and consume c_r per unit time. Let the rural individual supply the urban with an amount $N_r g_r$ of rural goods and receive for each unit of those goods θ units of urban products. As in chapter v, we have a demand function connecting g_r with θ, and we again put

$$g_r = g_0 - \frac{b}{\theta}. \tag{20}$$

Putting

$$\varepsilon_u = p_u - c_u; \quad \varepsilon_r = p_r - c_r; \quad \varepsilon = \varepsilon_r - \varepsilon_u; \tag{21}$$

and using for p_r the expression (3) and again considering the case $\varepsilon_u > 0$ with $p_u = \text{Const.}$, we have:

$$\varepsilon_u = \text{Const} > 0; \quad \varepsilon_r = \frac{a_1}{N_r + a_2} - c_r;$$
$$\varepsilon = \frac{a_1}{N_r + a_2} - c'; \quad c' = c_r + \varepsilon_u. \tag{22}$$

We now have for the rate of change of the total accumuluated goods for the urban and rural population:

$$\frac{dW_u}{dt} = N_u\varepsilon_u + N_r g_r - \theta N_r g_r,$$
$$\frac{dW_r}{dt} = N_r\varepsilon_r - N_r g_r + \theta N_r g_r. \tag{23}$$

Because of equations (1) and (20), (23) gives

$$\frac{dW_u}{dt} = N_u\varepsilon_u + (N - N_u)\left(g_0 - \frac{b}{\theta}\right)(1 - \theta);$$
$$\frac{dW_r}{dt} = N_r\varepsilon_r - N_r\left(g_0 - \frac{b}{\theta}\right)(1 - \theta). \tag{24}$$

The per capita rates v_u and v_r are:

$$v_u = \varepsilon_u + \frac{N - N_u}{N_u}\left(g_0 - \frac{b}{\theta}\right)(1 - \theta);$$
$$v_r = \varepsilon_r - \left(g_0 - \frac{b}{\theta}\right)(1 - \theta). \tag{25}$$

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CHAPTER XI

OUTLINE OF A THEORY OF THE SIZES OF CITIES

The considerations of the preceding chapter lead naturally to an inquiry not only into the relative sizes of the total urban and total rural populations, but also into the actual sizes of the individual cities.

In the first approximation we may neglect the distribution of city sizes and speak of the "average" size of cities as if all cities were of that average size. Next we shall treat the problem of the actual distribution of city sizes.

In treating the simple problem of the ratio of the urban to rural population, we considered that the amounts of goods produced per unit time by an individual in the city (p_u) and in the country (p_r) depend in general on both N_u and N_r, the total urban and rural population. We must now somewhat generalize this picture and consider that both p_u and p_r depend not only on N_u and N_r, but also on the average size of the cities or, what amounts to the same, on the number m of the cities. This is closer to reality than the original assumption, for the conditions of work in a community will depend in general on the size of that community.

We thus put

$$p_u = f_u(N_u, N_r, m); \quad p_r = f_r(N_u, N_r, m). \tag{1}$$

By the same argument as before, we have for equilibrium

$$p_u = p_r. \tag{2}$$

Equation (2) together with

$$N_u + N_r = N, \tag{3}$$

where N is the total population, determines N_u and N_r as functions of N and m; thus

$$N_u = F_u(N, m), \quad N_r = F_r(N, m). \tag{4}$$

Substituting equations (4) into (1), we find:

$$p_u = p_u(N, m); \quad p_r = p_r(N, m). \tag{5}$$

If each individual tends to choose or modify his surroundings in such a way as to make his p_u (or p_r) as large as possible, then all in-

dividuals will aggregate in cities of such a size which makes p_u and p_r a maximum. Hence we have for the determination of m:

$$\frac{dp_u}{dm} = 0 \quad \text{or} \quad \frac{dp_r}{dm} = 0. \tag{6}$$

Because of equation (2) either of the relations (6) leads to the same expression:

$$m = f(N). \tag{7}$$

Equations (4) and (7) determine N_u, N_r and m as functions of the total population.

Thus the determination of m in each individual case reduces to the determination of the functions f_u and f_r in equations (1). By making different plausible assumptions about these functions, we shall obtain different expressions for m. If $m = f(N)$ is known from observations, we may attempt to determine backwards the form of f_u and f_r. An important factor which will determine f_u and f_r is the development of means of transportation. The larger the m, and therefore the smaller the size of the cities, the lesser distance must be covered in transporting to the city rural supplies which influence the value of p_u. Similarly, the ease of supply of the rural population with urban products affects the value of p_r.

We may, for instance, put:

$$\begin{aligned} p_u &= a_1 - a_2 N_u + (a_3 + a_4 m) N_r; \\ p_r &= b_1 - b_2 N_r + (b_3 + b_4 m) N_u; \end{aligned} \tag{8}$$

where a_i and b_i are parameters. The coefficients a_4 and b_4 may be taken, for instance, as proportional to the total length of railroads and highways per square mile. We easily find

$$\frac{N_u}{N} = A - \frac{B}{N}, \tag{9}$$

where A and B are functions of m. The complete solution of the problem, using expression (8), is elementary but rather cumbersome.

Let us now consider the distribution of city sizes. We shall now consider any community as "city" and drop the distinction between urban and rural population. Let N_i be the total number of people inhabiting all cities of size (population) n_i. We shall have, in general, for the amount of goods produced per person in such cities:

$$p_i = f(n_i, N_i). \tag{10}$$

It may be questioned why p_i should depend explicitly on N_i. In view of interaction of different industries, the productivity p_i per person would depend on the distance between cities, those distances affecting the ease of transport of different products from city to city. Those distances, however, are determined by the number of the cities. Therefore, quite generally, we may assume relation (10).

For the equilibrium we have:

$$p_1 = p_2 = \cdots = \text{Const.} = p\,. \tag{11}$$

Hence

$$f(n_i\,, N_i) = p\,. \tag{12}$$

Equation (12) gives us N_i as a function of n_i and p:

$$N_i = \bar{f}(n_i\,, p)\,. \tag{13}$$

The quantity p is determined by the requirement

$$\sum N_i = N\,, \tag{14}$$

or

$$\sum \bar{f}(n_i\,, p) = N\,. \tag{15}$$

For very large numbers we may use integrals instead of sums and determine p from

$$\int_0^\infty \bar{f}(n\,, p)\; dn = N\,, \tag{16}$$

while equation (13) now reads

$$N(n) = \bar{f}(n\,, p)\,. \tag{17}$$

As an illustration we may consider such an expression as

$$p_i = a_i - b_i N_i\,; \tag{18}$$

where

$$a_i = f_1(n_i)\,; \quad b_i = f_2(n_i)\,. \tag{19}$$

We then find:

$$N(n) = \frac{f_1(n) - p}{f_2(n)}\,, \tag{20}$$

the value of p being determined by

$$\int_0^\infty \frac{f_1(n) - p}{f_2(n)}\, dn = N\,. \tag{21}$$

As another illustration we may consider

$$p_i = a_i - b_i N_i + \left(c_i + \sum \frac{N_i}{n_i}\right)(N - N_i);$$
$$a_i = f_1(n_1); \quad b_i = f_2(n_i); \quad c_i = f_3(n_i). \tag{22}$$

Since $\sum(N_i/n_i)$ is the total number of cities, this expression corresponds to expression (8), mentioned above. For the continuous case we have:

$$p = f_1(n) - f_2(n)N(n) + Nf_3(n) + N\int_0^\infty \frac{N(n)}{n}dn$$
$$- N(n)f_3(n) - N(n)\int_0^\infty \frac{N(n)}{n}dn. \tag{23}$$

Putting

$$K = \int_0^\infty \frac{N(n)}{n}dn, \tag{24}$$

K being the total number of cities, we find from equation (23),

$$N(n) = \frac{f_1(n) + Nf_3(n) + NK - p}{f_2(n) + f_3(n) + K}. \tag{25}$$

Introducing equation (25) into (24) we find a relation between p and K, thus

$$K = K(p), \tag{26}$$

so that equation (25) now becomes

$$N(n) = \frac{f_1(n) + Nf_3(n) + NK(p) - p}{f_2(n) + f_3(n) + K(p)}. \tag{27}$$

The value of p is again determined from

$$\int_0^\infty N(n)\,dn = N. \tag{28}$$

CHAPTER XII

THE DISTRIBUTION OF CITY SIZES

We shall now consider a different approach to the problem discussed in chapter xi. Before we do that, however, we shall briefly discuss the possible relation of the previous theory to available data.

G. K. Zipf[1] has found an interesting relation between the sizes of cities and their rank according to size. If we rank all cities consecutively in order of decreasing size and denote the population of a city of rank r by $n(r)$, then, according to Zipf, for many countries we have the relation

$$n(r) = \frac{C}{r}, \tag{1}$$

where C is a constant. By considering data for the United States and for Canada at different times, Zipf finds that relation (1) did not hold in the past, but has been gradually approached. He generalizes this by postulating that relation (1) is characteristic of a stable society, and is reached as a society passes from less stable to more stable configurations. In our opinion such a postulate seems to lack either empirical or theoretical evidence. It would therefore be of great interest if such a postulate could be obtained as a deduction from a rational theory. This would throw light on the mechanism underlying the simple relation (1).

Inasmuch as the distribution functions studied by us previously do not involve the rank order of the cities, we shall first investigate how Zipf's results can be translated into our notations.

Denoting by $r(n)$ the rank of a city having a population n, equation (1) may also be written

$$r(n) = \frac{C}{n}. \tag{2}$$

Let $\bar{N}(n)\delta n$ be the number of cities whose population lies between n and $n + \delta n$, δn being a small but finite quantity. Consider instead of equation (2) any arbitrary function $r(n)$ which must, however, decrease with increasing n. For very large values of $r(n)$, the number of cities $N(n)\delta n$ will be large even for small δn. Since $r(n)$ increases discontinuously and always in steps of one, therefore

the increment $-\delta r(n)$, which corresponds to the increment δn, represents actually the number of ranks, and therefore the number of cities, in the interval δn. Hence

$$\bar{N}(n)\ \delta n = -\delta r(n), \tag{3}$$

or

$$\bar{N}(n) = -\frac{\delta r(n)}{\delta n}. \tag{4}$$

For very large r's we can take a very small δn, and in the limit we shall then find

$$\bar{N}(n) = -\frac{dr}{dn}. \tag{5}$$

The notion of a continuous distribution function $\bar{N}(n)$ breaks down for very large values of n, for there are only a few very large cities in each country. Nevertheless, far away from the tail end of the curve, the function $\bar{N}(n)$ may be practically determined. The rank order notation has the advantage of covering the whole range of sizes.

The function $\bar{N}(n)$ is connected with the function $N(n)$ of chapter xi by the relation

$$\bar{N}(n) = \frac{N(n)}{n}. \tag{6}$$

Hence, if $r(n) = C/n$, then

$$\bar{N}(n) = \frac{C}{n^2};\quad N(n) = \frac{C}{n} = r(n). \tag{7}$$

Denote by N the total population

$$N = \int_0^\infty n\bar{N}(n)\,dn = \int_0^\infty N(n)\,dn. \tag{8}$$

A distribution function $N(n)$ such as the one given by equation (7) could be readily obtained from our previous theory by putting in chapter xi:

$$f(n, N) = AnN, \tag{9}$$

where A is a constant. That would give us, according to equation (17) of chapter xi:

$$N(n) = \frac{p}{An}, \tag{10}$$

p being determined as before from relation (8) of that chapter.

Equation (9) means that the per capita production of goods is

proportional to both n and N, a relation that does not seem to be very plausible. Because of relation (6) this would imply that the total production is proportional to $n^3 \bar{N}^2 (n)$.

The simple theory developed before does not seem therefore to account for the relation found by Zipf. Inasmuch as not all countries follow Zipf's relation, and inasmuch as the above mentioned theory may be modified and generalized, we should not yet discard it altogether. In a complex problem like this, in which many factors probably enter, it may be admissible to discuss *in abstracto* different *conceivable* theoretical cases without first worrying about actual data. A thorough classification of the theoretical possibilities may later prove a help in deciding which of those possibilities or their combinations can actually be applied to observations. It is in this spirit of an abstract theoretical study that we shall discuss an alternative approach to the problem without prejudice to other possible approaches.

It is natural to attempt to connect the formation of cities with the presence of active groups in a society. A city may originate as an administrative center, in which case its formation will be closely related with the activity of an administrative active group, which we shall denote as group I. A city may also originate as a trade or industrial center, in which case it will be associated with another active group, an organizing one, which we previously denoted as group II. The stronger the administrative group I, the larger we may expect the principal city to be. On the other hand, a strong industrial group will result in the formation of large industrial cities. It is therefore natural to inquire whether the distribution function of city sizes may not be connected with the distribution function of the gradation of different types of activities within the population.

Since this is a purely theoretical study, we shall not specify here what particular characteristics we do consider. We shall simply denote the measure of that characteristic by x and consider that in a population of N individuals the characteristic x is distributed according to some function $\mathrm{N}(x)$. In whatever physical or psychophysical units we measure the quantity x, we shall choose our units so that the maximum value of x in a given group is equal to 1. We then have:

$$N = \int_0^1 \mathrm{N}(x)\, dx . \tag{11}$$

We have seen previously that such a group may break up into several smaller groups if each individual associates only with those individuals whose x's are not too remote from his own. Equations for determination of the size of those classes have been given previously

(chapters i, ii). We thus find that the whole population is divided into $n+1$ groups, whose x's lie between $1-x_1$, x_1-x_2, x_2-x_3, $\cdots$ x_n-0, with $x_k > x_{k+1}$. The total amount of x in a group r is given by

$$X(r) = \int_{x_r}^{x_{r-1}} x\mathrm{N}(x)\, dx\,. \tag{12}$$

The corresponding "populations" are given by

$$N(r) = \int_{x_r}^{x_{r-1}} \mathrm{N}(x)\, dx\,. \tag{13}$$

Suppose now that these groups will be segregated spatially. They will thus form separate communities of different sizes.

Let x denote the measure of any special ability, such as executive, business, literary, etc., and consider the case where $\mathrm{N}(x)$ will be decreasing with increasing x. The functions $N(r)$ and $X(r)$ may either decrease or increase with r. It is clear that we cannot set $n(r)$ proportional to $N(r)$ if the latter increases, for the largest community will be composed of individuals with the smallest x. If, however, $N(r)$ and $X(r)$ decrease with increasing r, we may consider the following situation. The rth group of $N(r)$ individuals may form a nucleus around which a number $n'(r)$ of individuals of the lowest x gather to perform any activity, directed by the $N(r)$ individuals, and thus form a community of $n(r) = N(r) + n'(r)$ individuals. We thus assume that the class of lowest x's, namely that lying in the interval $(x_n, 0)$, is entirely passive. This class is also the most numerous one. The simplest assumption we may make about $n'(r)$ is that it is proportional to $X(r)$. We then have

$$n(r) = N(r) + \alpha X(r). \tag{14}$$

If $N(r) << \alpha X(r)$, then we have approximately

$$n(r) = \alpha X(r). \tag{15}$$

Thus $n(r)$ would be determined by $\mathrm{N}(x)$ and we may investigate what form of $\mathrm{N}(x)$ will give a prescribed $n(r)$, as for example, expression (1).

Although such a case presents some theoretical interest, it hardly can be applied to actual cases for it implies that the active group of each community has a different range of x's, the ranges for different cities never overlapping.

We may consider a somewhat more realistic assumption which is free from the above shortcoming. Let the group $N(1)$ form a nu-

cleus around which there will be gathered $n'(1) = \alpha X(1)$ individuals, but let those $n'(1)$ individuals be picked up from all the $N^{(1)} = N - N(1)$ individuals that are left outside of the group $N(1)$. Furthermore, let the contribution of individuals with a given x to $\alpha X(1)$ be in proportion to the frequency with which those individuals occur. We have

$$N^{(1)} = \int_0^{x_1} \mathrm{N}(x)\, dx\,. \tag{16}$$

The distribution function $N^{(2)}(x)$ of the individuals left after the $\alpha X(1)$ individuals have been subtracted from $N^{(1)}$ is equal, with the above assumption, to

$$N^{(2)}(x) = \left(1 - \frac{\alpha X(1)}{N^{(1)}}\right) \mathrm{N}(x)\,. \tag{17}$$

In other words, each class has lost a fraction $\alpha X(1)/N^{(1)}$ of individuals.

Now the group whose x lies between x_1 and x_2 has only

$$\bar{N}^{(2)} = \int_{x_2}^{x_1} N^{(2)}(x)\, dx \tag{18}$$

individuals, and their total x is equal to

$$X'(2) = \int_{x_2}^{x_1} x N^{(2)}(x)\, dx\,. \tag{19}$$

This group will gather around it $\alpha X'(2)$ individuals from the $N^{(2)}$ individuals left outside of the group. We have

$$N^{(2)} = \int_0^{x_2} N^{(2)}(x)\, dx\,. \tag{20}$$

The distribution function of the remaining individuals is given by

$$N^{(3)}(x) = \left(1 - \frac{\alpha X'(2)}{N^{(2)}}\right) N^{(2)}(x)\,, \tag{21}$$

and

$$X'(3) = \int_{x_3}^{x_2} x N^{(3)}(x)\, dx\,. \tag{22}$$

Thus we can consecutively calculate $N^{(1)}$, $\bar{N}^{(1)}$, $N^{(2)}$, $\bar{N}^{(2)}$, as well as $X(1), X'(2), X'(3), \cdots$, etc., and thus find $n(r)$ from

$$n(r) = \bar{N}^{(r)} + \alpha X'(r)\,. \tag{23}$$

This scheme may be generalized further by considering that as the distribution functions $N^{(i)}(x)$ change step by step, so will the set $x_1, x_2, \cdots, x_n$ change, according to equations developed previously.

The difficulty with actual calculation of such expressions lies in the circumstance that even the simplest forms of $\mathrm{N}(x)$ lead to transcendental equations for $x_1, x_2, \cdots, x_n$, which do not admit of closed solutions. In order to get an idea as to how such expressions as (12), (13), (14) and (23) behave, we shall make a very crude approximation and consider all intervals $x_i - x_{i+1}$ as equal to a small constant Δ:

$$x_i - x_{i+1} = \Delta . \tag{24}$$

For very small values of Δ an approximate expression for $X(r)$ and $N(r)$ is easily obtained. We may put approximately

$$X(r) = \int_{1-r\Delta}^{1-(r-1)\Delta} x\mathrm{N}(x)\, dx = \Delta x\mathrm{N}(x) \tag{25}$$

and

$$N(r) = \int_{1-r\Delta}^{1-(r-1)\Delta} \mathrm{N}(x)\, dx = \Delta\mathrm{N}(x). \tag{26}$$

Remembering that for the rth group x is approximately equal to $1 - r\Delta$, we find:

$$X(r) = \Delta(1 - r\Delta)\mathrm{N}(1 - r\Delta), \quad N(r) = \Delta\mathrm{N}(1 - r\Delta). \tag{27}$$

We shall consider here as a theoretically interesting case an approximate distribution function

$$\mathrm{N}(x) = Ax^{-\nu}, \quad \nu > 0 , \tag{28}$$

which is suggested by Pareto's law. The relation (28) cannot hold physically for $x = 0$. Most likely $\mathrm{N}(0) = 0$, but it may also be that $\mathrm{N}(0) > 0$. Except for exceedingly small values of x the approximation may be very good. The exact expression for $\mathrm{N}(x)$ should satisfy relation (11), which also determines the constant A.

For very small values of Δ we have from equations (27)

$$X(r) = A\Delta(1 - r\Delta)^{1-\nu}; \quad N(r) = A\Delta(1 - r\Delta)^{-\nu} . \tag{29}$$

If $X(r)$ is always to decrease with increasing r, we must have

$$0 < \nu < 1 . \tag{30}$$

In the following we shall always consider that the restriction (30) is satisfied.

The exact expressions for $X(r)$ and $N(r)$ are obtained from equa-

tions (12) and (13):

$$X(r) = \frac{A}{2-\nu}\{[1-(r-1)\Delta]^{2-\nu} - [1-r\Delta]^{2-\nu}\};$$
$$N(r) = \frac{A}{1-\nu}\{[1-(r-1)\Delta]^{1-\nu} - [1-r\Delta]^{1-\nu}\}. \tag{31}$$

We shall now derive an explicit form for expression (23) based on (24) and (28). We have

$$\bar{N}^{(1)} = A\int_{1-\Delta}^{1} x^{-\nu}\,dx = \frac{A}{1-\nu}[1-(1-\Delta)^{1-\nu}]; \tag{32}$$

$$N^{(1)} = A\int_{0}^{1-\Delta} x^{-\nu}\,dx = \frac{A}{1-\nu}(1-\Delta)^{1-\nu}; \tag{33}$$

$$X'(1) = X(1) = A\int_{1-\Delta}^{1} x^{1-\nu}\,dx = \frac{A}{2-\nu}[1-(1-\Delta)^{2-\nu}]. \tag{34}$$

Hence

$$N^{(2)}(x) = \left(1-\frac{\alpha X(1)}{N^{(1)}}\right)\mathrm{N}(x)$$
$$= \left(1-\alpha\frac{[1-(1-\Delta)^{2-\nu}](1-\nu)}{(1-\Delta)^{1-\nu}(2-\nu)}\right)Ax^{\nu}; \tag{35}$$

$$X'(2) = \int_{1-2\Delta}^{1-\Delta} xN^{(2)}(x)\,dx =$$
$$\left(1-\alpha\frac{[1-(1-\Delta)^{2-\nu}](1-\nu)}{(1-\Delta)^{1-\nu}(2-\nu)}\right)\frac{A}{2-\nu}[(1-\Delta)^{2-\nu}-(1-2\Delta)^{2-\nu}]. \tag{36}$$

Define

$$a = \alpha\frac{1-\nu}{2-\nu}; \tag{37}$$

and

$$f(r) = \left(1-a\frac{1-(1-\Delta)^{2-\nu}}{(1-\Delta)^{1-\nu}}\right)\left(1-a\frac{(1-\Delta)^{2-\nu}-(1-2\Delta)^{2-\nu}}{(1-2\Delta)^{(1-\nu)}}\right)$$
$$\cdots\left(1-a\frac{[1-(r-1)\Delta]^{2-\nu}-[1-r\Delta]^{2-\nu}}{(1-r\Delta)^{1-\nu}}\right). \tag{38}$$

We shall now prove that in general

$$N^{(r)}(x) = Af(r-1)x^{-\nu} ; \tag{39}$$

$$X'(r) = \frac{A}{2-\nu} f(r-1)\{[1-(r-1)\Delta]^{2-\nu} - [1-r\Delta]^{2-\nu}\}. \tag{40}$$

Expressions (39) and (40) hold for $r = 1$ and $r = 2$, as we have seen. We shall prove that if they hold for r, they hold for $r+1$.

From equation (39) we have

$$N^{(r)} = \int_0^{1-r\Delta} N^{(r)}(x)\, dx = \frac{A}{1-\nu} f(r-1)(1-r\Delta)^{1-\nu}, \tag{41}$$

$$N^{(r+1)}(x) = \left(1 - \frac{aX'(r)}{N^{(r)}}\right) N^{(r)}(x)\, dx =$$

$$\left(1 - a\frac{1-\nu}{2-\nu} \frac{[1-(r-1)\Delta]^{2-\nu} - [1-r\Delta]^{2-\nu}}{(1-r\Delta)^{1-\nu}}\right) f(r-1)Ax^{-\nu}. \tag{42}$$

Because of expressions (38) and (39), equation (42) may be written

$$N^{(r+1)}(x) = Af(r)\, x^{-\nu}. \tag{43}$$

We have further

$$X'(r+1) = \int_{1-(r+1)\Delta}^{1-r\Delta} xN^{(r+1)}(x)\, dx,$$

which, because of equation (43), may be written

$$X'(r+1) = \frac{A}{2-\nu} f(r)\{[1-r\Delta]^{2-\nu} - [1-(r+1)\Delta]^{2-\nu}\}. \tag{44}$$

This proves expressions (39) and (40).

We have

$$\bar{N}(r) = \int_{1-r\Delta}^{1-(r-1)\Delta} N^{(r)}(x)\, dx = \frac{A}{1-\nu} f(r-1)\{[1-(r-1)\Delta]^{1-\nu} - [1-r\Delta]^{1-\nu}\}. \tag{45}$$

Hence, introducing expressions (40) and (45) into (23), we have

$$n(r) = \frac{A}{1-\nu} f(r-1)\{[1-(r-1)\Delta]^{1-\nu} - [1-r\Delta]^{1-\nu}\} + \frac{aA}{2-\nu} f(r-1)\{[1-(r-1)\Delta]^{2-\nu} - [1-r\Delta]^{2-\nu}\}. \tag{46}$$

The expression $f(r)$ simplifies considerably for very small values of Δ. Putting for any k:

$$k\Delta = y \tag{47}$$

we have

$$[1-(k-1)\Delta]^{2-\nu} - [1-k\Delta]^{2-\nu} = [1-(y-\Delta)]^{2-\nu} - [1-y]^{2-\nu}; \tag{48}$$

For very small values of Δ, this is equal to

$$-\Delta \frac{d}{dy}(1-y)^{2-\nu} = (2-\nu)\Delta(1-y)^{1-\nu} = (2-\nu)\Delta(1-k\Delta)^{1-\nu}. \tag{49}$$

Hence, because of expressions (48), (37) and (49),

$$1 - a\frac{[1-(k-1)\Delta]^{2-\nu} - [1-k\Delta]^{2-\nu}}{(1-k\Delta)^{1-\nu}} = 1-\alpha(1-\nu)\Delta; \tag{50}$$

and, because of expression (38):

$$f(r) = [1-\alpha(1-\nu)\Delta]^{r}. \tag{51}$$

Introducing this into equation (40) and transforming the expression in braces of (40) according to expressions (48) and (49), we have:

$$X'(r) = A\Delta(1-r\Delta)^{1-\nu}[1-\alpha(1-\nu)\Delta]^{r-1}. \tag{52}$$

Since physically we must have

$$0 << 1-\alpha(1-\nu)\Delta < 1, \tag{53}$$

therefore, comparing expression (52) with (29), we see that $X'(r)$ decreases with r more rapidly than $X(r)$. This should be physically so because while $X(r)$ refers to the group formed of all individuals who have an x between $1-r\Delta$ and $1-(r-1)\Delta$, $X'(r)$ refers to the group formed by those individuals within that interval that were left over after subtracting the amount which contributed to the $r-1$ preceding n's. Hence $X(r) > X'(r)$.

It should be noticed that $X(r)$ decreases with r less rapidly than $1/r$, but $X'(r)$ decreases more rapidly than $1/r$.

By a similar procedure, using expressions (48), (49) and (51), we obtain from equation (45) for very small values of Δ:

$$\bar{N}^{(r)} = A\Delta(1-r\Delta)^{-\nu}[1-\alpha(1-\nu)\Delta]^{r-1}. \tag{54}$$

Introducing expressions (52) and (54) into equation (23) we find:

$$n(r) = A\Delta[1-\alpha(1-\nu)\Delta]^{r-1}\{(1-r\Delta)^{-\nu}+\alpha(1-r\Delta)^{1-\nu}\}. \qquad (55)$$

The variation of $n(r)$ with r is rather complicated. For small values of r the term $[1 - \alpha(1 - \nu)\Delta]^{r-1}$ decreases more rapidly than $(1 - r\Delta)^{-\nu}$ increases. Therefore for small values of r the quantity $\overline{N}^{(r)}$ decreases, but less rapidly than $X'(r)$. The quantity $n(r)$ decreases also. However, since for $r = 1/\Delta$, the term $(1-r\Delta)^{-\nu}$ becomes infinite, $n(r)$ has a minimum for some value of $r = r_m$. Since by definition r is the rank order of *decreasing* sizes, such a situation would be physically absurd. This difficulty may be avoided, however, by the following consideration. Equation (55) is based on the approximate expressions (52) and (54) which cannot be applied for values of r that are close to $1/\Delta$. It must be remembered that r varies from 1 to $1/\Delta$ only. If the parameters in equation (55) can be chosen so that the value r_m for which $n(r)$ has a minimum is greater than $1/\Delta - 1$, then the above difficulty will be avoided. We shall now prove that this can be done.

Denote

$$\gamma = 1-\alpha(1-\nu)\Delta; \quad 0<\gamma<1. \qquad (56)$$

Equation (55) now becomes:

$$n(r) = A\Delta\gamma^{r-1}\{(1-r\Delta)^{-\nu}+\alpha(1-r\Delta)^{1-\nu}\}. \qquad (57)$$

We have

$$\frac{dn(r)}{dr} = A\Delta\gamma^{r-1}(1-r\Delta)^{-\nu}[\nu\Delta(1-r\Delta)^{-1}+\log\gamma-\alpha(1-\nu)\Delta + \alpha(1-r\Delta)\log\gamma]. \qquad (58)$$

The value r_m is defined by $dn(r)/dr = 0$ or

$$\nu\Delta + (1-r\Delta)[\log\gamma-\alpha(1-\nu)\Delta]+\alpha(1-r\Delta)^2\log\gamma = 0. \qquad (59)$$

Introduce the new variable

$$z = 1-r\Delta; \quad z_m = 1-r_m\Delta. \qquad (60)$$

We then have

$$z_m^2\,\alpha\log(1/\gamma)+z_m[\log(1/\gamma)+\alpha(1-\nu)\Delta]-\nu\Delta = 0. \qquad (61)$$

Equation (61) gives

$$z_m = [1/2\alpha\log(1/\gamma)]\{-[\log(1/\gamma)+\alpha(1-\nu)\Delta] + \sqrt{[\log(1/\gamma)+\alpha(1-\nu)\Delta]^2+4\alpha\nu\Delta\log(1/\gamma)}\}. \qquad (62)$$

The positive sign must be taken before the square root because $z_m > 0$.

If we wish to have $r_m < 1/\Delta - 1$, then we must have

$$z_m > \Delta . \tag{63}$$

Inequality (63) will be satisfied if γ is made sufficiently small. To show this we make use of expressions (56) and write equation (62) thus:

$$z_m = [1/2\alpha \log(1/\gamma)] \left\{ - [\log(1/\gamma) + 1 - \gamma] + \sqrt{[\log(1/\gamma) + 1 - \gamma]^2 + \frac{4(1-\gamma)\nu}{1-\nu} \log(1/\gamma)} \right\} . \tag{64}$$

As γ becomes very small, log $(1/\gamma)$ becomes very large. Thus we may neglect $1 - \gamma$ in the expression in brackets, and also neglect γ as compared with 1. We then have:

$$z_m = [1/2\alpha \log(1/\gamma)] \left\{ - \log\frac{1}{\gamma} + \log(1/\gamma) \sqrt{1 + \frac{4\nu}{1-\nu} \frac{1}{\log(1/\gamma)}} \right\} . \tag{65}$$

The quantity $1/\log(1/\gamma)$ being now very small, we may expand the expression under the square root sign, keeping only linear terms. We thus find

$$z_m = [1/2\alpha \log(1/\gamma)] \frac{2\nu}{1-\nu} . \tag{66}$$

By making γ sufficiently small, we can always satisfy inequality (63).

But a small γ means a sufficiently large α. Hence inequality (63) may be satisfied by taking α sufficiently large, though not large enough to make γ negative.

Thus with a proper choice of α, $n(r)$, as given by equation (55), will decrease monotonically with r within the range $1 \leqq r \leqq 1/\Delta - 1$. We may now investigate under what conditions, if any, $n(r)$ will vary within a wide range *approximately* as $1/r$, so as to satisfy relation (1). This may be done by expanding the right side of equation (55) into a power series and comparing the coefficients of the expansion with those of the expansion of $1/(\delta + r)$, where $\delta << 1$; for $1/(\delta + r)$ will for all practical purposes vary as $1/r$.

The general theory of the breaking up of a social group into classes, as developed in chapters i and ii, is based on the assumption

that only such individuals associate with each other for whom the difference $(x' - x)^2$ is less than a certain quantity Δ_0. We had as criterion

$$(x' - x)^2 < \Delta^2_0 . \tag{67}$$

We shall now consider a different criterion which is perhaps somewhat more realistic. We shall assume, namely, that it is not the difference $x' - x$ but the ratio x'/x that determines whether two individuals associate with each other or not. The plausibility of such an assumption is suggested by the following considerations.

An individual with an income of 100,000 dollars is likely to associate with another individual whose income is 75,000 dollars, but an individual with an income of 26,000 dollars is not likely to associate with an individual having an income of 1,000 dollars. The difference is the same in both cases but the ratios are different. Similarily, an executive or a politician who controls directly or indirectly 10,000 individuals, will associate with another one who controls 6,000 individuals, but an executive having control over 5,000 individuals will not associate with a foreman having control over 25 individuals.

Instead of inequality (67) we may now put

$$(\log x' - \log x)^2 < \Delta^2_0 , \tag{68}$$

and instead of using equation (9) of chapter ii, we shall determine x_1 from the equation

$$\int_{x_1}^{1} \int_{x_1}^{1} [(\log x' - \log x)^2 - \Delta_0^2] \, \mathrm{N}(x) \mathrm{N}(x') \, dx dx' = 0 . \tag{69}$$

Equations of similar form will determine x_2, x_3, etc.

We again run into the same difficulty as before, namely: the equations determining the x_i's are transcendental. Therefore we shall again use a very rough approximation, corresponding to equation (24) of the previous case. We shall put

$$x_i = \beta^i , \quad 0 < \beta < 1 . \tag{70}$$

For $\mathrm{N}(x)$ we shall again use (28).

We find now, with reference to expressions (12) and (13):

$$X(r) = A \int_{\beta^r}^{\beta^{r-1}} x^{1-\nu} \, dx = \frac{A(1-\beta^{2-\nu})}{2-\nu} \beta^{(2-\nu)(r-1)} ; \tag{71}$$

$$N(r) = A \int_{\beta^r}^{\beta^{r-1}} x^{-\nu} \, dx = \frac{A(1-\beta^{1-\nu})}{1-\nu} \beta^{(1-\nu)(r-1)} . \tag{72}$$

Now both $X(r)$ and $N(r)$ decrease monotonically with r, so that equation (14) can be used, giving

$$n(r) = A\left(\frac{1-\beta^{1-\nu}}{1-\nu}\beta^{(1-\nu)(n-1)} + \alpha\frac{1-\beta^{2-\nu}}{2-\nu}\beta^{(2-\nu)(n-1)}\right). \tag{73}$$

It is readily seen that $n(r)$ decreases much more rapidly than $1/r$.

We now calculate $X'(r)$, $N^{(r)}(x)$ and $\overline{N}^{(r)}$. We have

$$X'(1) = A\int_{\beta}^{1} x^{1-\nu}\,dx = \frac{A(1-\beta^{2-\nu})}{2-\nu}; \tag{74}$$

$$N^{(1)} = A\int_{0}^{\beta} x^{-\nu}\,dx = \frac{A\beta^{1-\nu}}{1-\nu}; \tag{75}$$

$$\begin{aligned} N^{(2)}(x) &= \left(1-\frac{\alpha X'(1)}{N^{(1)}}\right)N(x) \\ &= A\left(1-\alpha\frac{(1-\nu)(1-\beta^{2-\nu})}{(2-\nu)\beta^{1-\nu}}\right)x^{-\nu}; \end{aligned} \tag{76}$$

$$\begin{aligned} X'(2) &= \int_{\beta^2}^{\beta} xN^{(2)}(x)\,dx \\ &= \left(1-\alpha\frac{(1-\nu)(1-\beta^{2-\nu})}{(2-\nu)\beta^{1-\nu}}\right)\frac{A(1-\beta^{2-\nu})}{2-\nu}\beta^{2-\nu}. \end{aligned} \tag{77}$$

Define

$$b = \alpha\frac{(1-\nu)(1-\beta^{2-\nu})}{2-\nu}; \tag{78}$$

and

$$f_1(y) = (1-b\beta^{\nu-1})(1-b\beta^{\nu})(1-b\beta^{\nu+1})\cdots(1-b\beta^{y+\nu}). \tag{79}$$

We shall now prove that in general

$$\begin{aligned} N^{(r)}(x) &= Af_1(r-3)\,x^{-\nu}; \\ X'(r) &= \frac{Ab}{\alpha(1-\nu)}f_1(r-3)\,\beta^{(2-\nu)(r-1)}. \end{aligned} \tag{80}$$

We have from (80):

$$N^{(r)} = \int_0^{\beta^r} N^{(r)}(x)\,dx = \frac{A}{1-\nu} f_1(r-3)\beta^{r(1-\nu)}. \tag{81}$$

Hence

$$\begin{aligned} N^{(r+1)}(x) &= \left(1 - \frac{aX'(r)}{N^{(r)}}\right) N^{(r)}(x) = (1 - b\beta^{\nu+r-2}) N^{(r)}(x) \\ &= Af_1(r-2)x^{-\nu} \end{aligned} \tag{82}$$

and

$$\begin{aligned} X'(r+1) &= \int_{\beta^{r+1}}^{\beta^r} xN^{(r+1)}(x)\,dx = f_1(r-2)\,\frac{A(1-\beta^{2-\nu})}{2-\nu}\,\beta^{r(2-\nu)} \\ &= \frac{Ab}{a(1-\nu)} f_1(r-2)\beta^{(2-\nu)r}. \end{aligned} \tag{83}$$

Since equation (81) holds for $r = 2$, it therefore holds for any r. We also have

$$\bar{N}^{(r)} = \int_{\beta^r}^{\beta^{r-1}} N^{(r)}(x)\,dx = \frac{A(1-\beta^{1-\nu})}{1-\nu} f_1(r-3)\beta^{(r-1)(1-\nu)}. \tag{84}$$

Introducing expressions (80) and (84) into (23) we find:

$$n(r) = \frac{A}{1-\nu} f_1(r-3)\;\beta^{(r-1)(1-\nu)}(1-\beta^{1-\nu}+b\beta^{r-1}). \tag{85}$$

The quantity $n(r)$ decreases monotonically with r, but more rapidly than $1/r$.

We may consider a more general assumption, namely that β varies from class to class. Denoting by $\beta_1, \beta_2, \cdots, \beta_i$ a sequence of numbers such that

$$0 < \beta_i < 1, \tag{86}$$

we may put

$$x_i = \beta_1\beta_2\cdots\beta_i. \tag{87}$$

In that case

$$X(r) = A\int_{\beta_1\beta_2\ldots\beta_r}^{\beta_1\beta_2\ldots\beta_{r-1}} x^{1-\nu}\,dx = \frac{A(1-\beta_r{}^{2-\nu})}{2-\nu}\,(\beta_1\cdots\beta_r)^{2-\nu}. \tag{88}$$

The quantity $X(r)$ decreases with r and by a proper choice of the se-

quence $\beta_1\beta_2 \cdots \beta_i$ it may be made to decrease as $1/r$. Equation (15) would then lead to equation (1). A similar assumption may be studied for the more general case involving $N^{(r)}$, $X'(r)$ and equation (23).

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CHAPTER XIII

SLOW VARIATIONS OF SOCIAL RELATIONS. THE MECHANISM OF HISTORY

The phenomena studied in chapters iii-viii are all essentially of a static nature. Given a definite distribution function of a characteristic of the population which determines the sizes of the different active classes and of the passive one, we obtain a definite behavior pattern for the population as a whole. This behavior pattern remains constant as long as the composition of the classes remains constant. Although we have studied in chapter iii and viii possible variations of the behavior pattern with respect to time, these variations are of short duration and principally determine the mechanisms of the relatively sudden transitions from one behavior pattern to another which occur when the composition of the active classes changes. But nothing has been said as yet concerning the causes which may produce such changes in the composition.

The considerations of chapter x also deal essentially with quasi stable phenomena, for the mechanism of mobility between urban and rural populations has not been considered and only equilibrium configurations have been treated. These equilibrium configurations slowly shift as the size or composition of the social group varies.

In the subsequent chapters we shall study the possible causes of variations in the composition of a social group with respect to time, and thus discuss long range variations of social structures. This naturally introduces us to a mathematical theory of history. The historical phenomena which can be treated mathematically in this fashion are *somewhat* restricted in scope. No one will ever hope to derive mathematically how many wives Henry VIII had, or how long Louis XIV reigned, or what kind of entertainment was current at his court, although this type of factual information does fill a not insignificant part of some history books. What mathematical history should be able to describe eventually are the changes with time in social relations, in forms of governments, legislation, art, philosophy, religion, etc.

In chapter i it was indicated that one possible cause of variation of sizes or relative strength of different active groups actually found in human history is due to the formation of closed hereditary social classes. Due to the dissimilarity of parents and offspring, such a he-

reditary closed class, which originally may be all composed of active individuals of a given type, will gradually be "thinned out" by passive individuals, born to active parents. This results in a general weakening of the activeness of the class as a whole. On the other hand, the passive class gains some active individuals and thus becomes less passive as a whole. In this case we have a change in social structure and therefore in the behavior pattern of a society as a whole, in spite of the fact that the actual total numbers of active and passive individuals, or at least their ratios, may remain unchanged. In other words, this is the case of a constant distribution function or, as we shall call it for brevity, of constant profile of the society. Some cases belonging to that group will be discussed in the next chapter.

There are possibilities, however, of actual variations of the profile of society with respect to time, as we have seen in chapter ii. Due to different birth rates and due to different possible kinds of interbreeding between individuals of different types, we may have aperiodic or periodic long range variations in the structure of society. The relative number of different types of active and passive individuals will fluctuate or otherwise vary in time. This will result in fluctuations of behavior, in other words, of the social relations of the whole population, although these fluctuations will not necessarily be periodical. The whole problem of social changes in a population thus becomes essentially a problem of biology of mutual interaction of different species, regarding each type of individual as a species. In sociology and history changes with respect to time in ideologies, morals, tastes, etc., are frequently discussed. A quantitative mathematical description of such abstract, almost intangible, things may well be impossible. The problem changes radically when we speak of the changes in the relative numbers of individuals professing given ideologies, morals, or tastes. A quantitative description of these changes is quite possible. We shall consider these changes as the fundamental mechanism underlying the phenomena of history and shall discuss some of them in subsequent chapters.

It must be emphasized that the possibility of a mathematical description of history does in no way depend on the picture which we adopted here; namely, the interaction of active and passive individuals. The assumption of a sharp distinction between actives and passives is of course an assumption made for the simplification of the mathematical treatment. As we have seen in chapter iv, a more rigorous treatment of continuous distributions leads to essentially the same results, at least in some simpler cases. The concept of individuals with different degrees of influence and of susceptibility to influence is realistic enough and can hardly be regarded as an assumption. On

the approximate treatment of that concept we base our mechanism of history. But if anyone does not agree at all with the picture of interaction of more active and less active groups, postulated here, it will still be possible to describe in mathematical terms changes in customs, ideologies, etc. We may, for instance, consider in equation (21) or (34) of chapter iii the quantities x_0 and y_0 not as the number of active individuals exhibiting behavior A and B, but purely formally, as some variables in terms of which the behavior of the whole society can be expressed by means of such equations as (21) or (34) of chapter iii.

CHAPTER XIV

CONSTANT PROFILE VARIATIONS OF SOCIETY

I

Let us now consider again, as in chapter iii, a population of N individuals, with x_0 active individuals I_A and y_0 active individuals I_B, the remainder being passive. Let inequality (8) of chapter iii be satisfied, so that all N' individuals of type II exhibit behavior A. By virtue of associations with similar individuals, all individuals of type I_A will form a social class, in a manner previously described. Since we are dealing with a simplified case where a uniformity of all individuals of a type is assumed, this social class will be composed, with the same approximation, of the x_0 individuals I_A. Similarly, there will be a social class of y_0 individuals of type I_B, and a social class of N' individuals II. The first social class (I_A) controls the behavior of the whole population. With the majority of the population exhibiting behavior A, the group I_B may either be prevented from activity B, or even entirely suppressed and destroyed.

Let us now study the variation of the composition of the controlling class with respect to time under the assumption made previously; that is, we take into account that of the total progeny of individuals I_A, only a fraction α will itself belong to a type I_A, the remainder belonging to type II. However, let the progeny of the first class associate only with the progeny of the same class. That is, instead of association by actual similarity, we shall have an association by the *similarity of the past generations*. As we have seen in chapter i, this results in an increase of individuals of type II in the first class. We shall call class A the social class composed *initially* of individuals of type I_A; we shall call class B the social class composed initially of individuals of type I_B; while the social class composed initially of individuals of type II will be referred to as class II.

Let n_A be the total number of individuals in the social class A. Let $n_A{}^A$ be the number of individuals in class A of the type I_A, $n_A{}^{II}$ be the number of individuals in class A of type II, and $n_A{}^B$ be the number of individuals in class A of type I_B. Similarly, we denote by n_{II} the total number of individuals in class II, by $n_{II}{}^{II}$ the number of individuals in class II of type II, by $n_{II}{}^A$ the number of individuals in class II

of type A, and by n_{II}^{B} the number of individuals in class II of type B. And, in a similar way, we define n_B, n_B^{A}, n_B^{II}, n_B^{B}. We have

$$N = n_A + n_{II} + n_B, \tag{1}$$

and

$$n_A = n_A^{A} + n_A^{II} + n_A^{B}, \quad n_B = n_B^{A} + n_B^{II} + n_B^{B},$$
$$n_{II} = n_{II}^{II} + n_{II}^{A} + n_{II}^{B}. \tag{2}$$

Consider here for simplicity the case in which the rate of increase of the population is proportional to the population. In many cases this is a good approximation to actual conditions.

Denoting by β_A, β_{II}, and β_B three coefficients of proportionality, we now have

$$\frac{dn_A}{dt} = \beta_A n_A; \quad \frac{dn_{II}}{dt} = \beta_{II} n_{II}; \quad \frac{dn_B}{dt} = \beta_B n_B. \tag{3}$$

In general, of course, we should take three different values β_A, β_{II}, and β_B. We consider here, only for the sake of simplicity, the case where

$$\beta_A = \beta_{II} = \beta_B = \beta. \tag{4}$$

A case of different birth and death rates for different classes will be discussed in chapter xvi.

Of all the *progeny* in any class, let the fraction α_A be of type I_A, the fraction α_B be of type I_B, the remainder $1 - \alpha_A - \alpha_B$ being of type II. Strictly speaking, these fractions should differ for different classes, but, again for simplicity, we consider them as the same for all classes. The more general case is treated in a similar way. We now have

$$\frac{dn_A^{A}}{dt} = \alpha_A \beta n_A; \quad \frac{dn_{II}^{A}}{dt} = \alpha_A \beta n_{II}; \quad \frac{dn_B^{B}}{dt} = \alpha_B \beta n_B, \tag{5}$$

and similar equations for the other n_i^{k}.

Equations (3) and (4), together with the consideration that initially we have $n_A = x_0$, $n_B = y_0$, $n_{II} = N'_0$, give for any value of $t > 0$:

$$n_A = x_0 e^{\beta t}; \quad n_{II} = N'_0 e^{\beta t}; \quad n_B = y_0 e^{\beta t}. \tag{6}$$

Substituting equations (6) into (5), we have:

$$\frac{dn_A^{A}}{dt} = \alpha_A \beta x_0 e^{\beta t}; \quad \frac{dn_{II}^{A}}{dt} = \alpha_A \beta N'_0 e^{\beta t}; \quad \frac{dn_B^{B}}{dt} = \alpha_B \beta y_0 e^{\beta t}, \tag{7}$$

with similar equations for other n_i^{k}.

Equations (7) give:

$$n_A{}^A = \alpha_A x_0 e^{\beta t} + C', \quad n_{II}{}^A = \alpha_A N'_0 e^{\beta t} + C'',$$
$$n_B{}^B = \alpha_B y_0 e^{\beta t} + C''', \tag{8}$$

C', C'', and C''' being integration constants. Since for $t = 0$, $n_A{}^A = x_0$, $n_{II}{}^A = 0$, $n_B{}^B = y_0$, therefore

$$x_0 = \alpha_A x_0 + C'; \quad \alpha_A N'_0 + C'' = 0; \quad y_0 = \alpha_B y_0 + C'''. \tag{9}$$

Hence

$$C' = x_0(1 - \alpha_A); \quad C'' = -\alpha_A N'_0; \quad C''' = y_0(1 - \alpha_B). \tag{10}$$

Introducing equations (10) into (8) gives:

$$n_A{}^A = x_0 + \alpha_A x_0 (e^{\beta t} - 1); \quad n_{II}{}^A = \alpha_A N'_0 (e^{\beta t} - 1);$$
$$n_B{}^B = y_0 + \alpha_B y_0 (e^{\beta t} - 1). \tag{11}$$

Comparison of equation (11) with (6) shows that for $t = \infty$

$$n_A{}^A = \alpha_A n_A; \quad n_{II}{}^A = \alpha_A n_{II}; \quad n_B{}^B = \alpha_B n_B. \tag{12}$$

Expressions (11) hold under the assumption that the principle of "hereditary classification" is enforced rigidly. In general we have a certain amount of "social mobility,"[1] so that individuals of class II may pass into class I. Let a fraction $\bar{\eta}$ of all individuals of type A born per unit time in class II pass into class I. Since per unit time altogether $\alpha_A \beta N'_0 e^{\beta t}$ [c.f. equation (7)] such individuals are born, we now have, instead of equation (7):

$$\frac{dn_A{}^A}{dt} = \alpha_A \beta x_0 e^{\beta t} + \bar{\eta} \alpha_A \beta N'_0 e^{\beta t};$$
$$\frac{dn_{II}{}^A}{dt} = (1 - \bar{\eta}) \alpha_A \beta N'_0 e^{\beta t}. \tag{13}$$

The equations for the other $n_i{}^k$ remain the same as before. Equations (13), with the same initial conditions as before, give

$$n_A{}^A = x_0 + \alpha_A (x_0 + \bar{\eta} N'_0)(e^{\beta t} - 1),$$
$$n_{II}{}^A = (1 - \bar{\eta}) \alpha_A N'_0 (e^{\beta t} - 1). \tag{14}$$

Now we have for $t = \infty$, instead of equations (12):

$$n_A{}^A = \left(1 + \frac{\bar{\eta} N'_0}{x_0}\right) \alpha_A n_A; \quad n_{II}{}^A = (1 - \bar{\eta}) \alpha_A n_{II}, \tag{15}$$

which reduces to equations (12) for $\bar{\eta} = 0$.

Let us consider possible effects of such changes in the composition of the different social classes, as described by the above equations for sufficiently small mobility $\bar{\eta}$. At the beginning, that is at $t = 0$, class A was the controlling class, because it was all composed of individuals of type I_A and their number satisfied the "controlling inequality" (8) of chapter iii. Class A continues to control the rest of the population by a sort of hysteresis, which may be in this case appropriately called tradition, although the relative number of individuals of type I_A in class A, the only ones who can *actively* control, gradually decreases. On the other hand, class II acquires a certain fraction of individuals of type I_A whose behavior is controlled, again by tradition, by those of class A, although they are actually of a controlling type themselves. Moreover, there is class B, which originally was too small and therefore did not gain control, but whose relative size might have increased, either because $\alpha_B > \alpha_A$, or because the asymptotic value of $n_B{}^B/N$ for $t = \infty$ is greater than the initial value.

Let us first consider the case in which the total number of individuals of type I_B remains negligible. This will happen when α_B is very small. Then the individuals of type I_A, belonging to the class II by accident of birth, will try to gain control over the whole population. They will actually gain control when an inequality, corresponding to (8) of chapter iii, will become satisfied. In this inequality we must now substitute $n_{II}{}^A$ for x_0, and $n_A{}^A$ for y_0, since it is those two groups that now dispute control. The control now will consist not of imposing on the population the behavior A (since it is already imposed); but while individuals I_A of class I will seek to impose obedience to them (behavior A'), individuals I_A of class II will seek to impose obedience to them (behavior A''). For N in inequality (8) of chapter iii, we must introduce the total population at the time t, which because of equations (1) and (6) equals

$$N(t) = N_0 \, e^{\beta t} . \tag{16}$$

Introducing again the coefficients a, a_0, c_0 of chapter iii and letting c_0 refer to individuals of type I_A in class A, a_0 to those of type I_A in class II, and a to those of type II in class II, and introducing the notations

$$\mu_1 = \frac{a}{a_0 + a}; \quad \mu_2 = \frac{c_0 - a}{a_0 + a}, \tag{17}$$

the condition for the taking over of control by individuals of type I_A in class II becomes

$$n_{II}^A > \mu_1 N(t) + \mu_2 n_A^A \,. \tag{18}$$

Introducing for n_{II}^A, n_A^A and $N(t)$ their values from expressions (14) and (16), we obtain

$$\begin{aligned}(1-\bar{\eta})\alpha_A N'_0(e^{\beta t}-1) > \mu_1 N_0 e^{\beta t} \\ + \mu_2[x_0 + \alpha_A(x_0+\bar{\eta}N'_0)(e^{\beta t}-1)].\end{aligned} \tag{19}$$

The time t^* when the change in control will happen is obtained by substituting for the inequality sign a sign of equality and solving the resulting equation for t. This gives, after rearrangements:

$$t^* = \frac{1}{\beta}\log\frac{(1-\bar{\eta})\alpha_A N'_0 + \mu_2[x_0 - \alpha_A(x_0+\bar{\eta}N'_0)]}{(1-\bar{\eta})\alpha_A N'_0 - \mu_1 N_0 - \mu_2\alpha_A(x_0+\bar{\eta}N'_0)}. \tag{20}$$

Since for $t=0$ inequality (19) does not hold, and both sides of (19) are exponentially increasing, therefore, if (19) is to hold for positive values of t, it must hold for $t=\infty$, which gives

$$(1-\bar{\eta})\alpha_A N'_0 > \mu_1 N_0 + \mu_2\alpha_A(x_0+\bar{\eta}N'_0). \tag{21}$$

Inequality (21) shows that the denominator of equation (20) is positive. Therefore, in order that t^* should be real and positive, the numerator must be greater than the denominator. This gives

$$\begin{aligned}(1-\bar{\eta})\alpha_A N'_0 + \mu_2[x_0 - \alpha_A(x_0+\bar{\eta}N'_0)] \\ > (1-\bar{\eta})\alpha_A N'_0 - \mu_1 N_0 - \mu_2\alpha_A(x_0+\bar{\eta}N'_0),\end{aligned} \tag{22}$$

which is equivalent to

$$\mu_1 N_0 + \mu_2 x_0 > 0\,, \tag{23}$$

which is always satisfied. Hence t^* is always real and positive if inequality (19) holds at all. $\bar{\eta}$ must not be too large, for when $\eta = 1$ or is near unity, practically all individuals of type I_A will be in class A. When $\bar{\eta}=1$, inequality (21) cannot be satisfied.

With

$$\mu_1 = 10^{-2};\quad \mu_2 = 1;\quad x_0 = 3\times 10^5;$$
$$\bar{\eta} = 10^{-2};\quad N_0 = N'_0 = 10^7;$$
$$\alpha_A = 3\times 10^{-2};\ \beta = 10^{-2}\ \text{year}^{-1}$$

we find from equation (20): $t^* \backsim 10^2\text{-}10^3$ years.

A somewhat different situation will develop when the class B is originally rather large, so that while inequality (8) of chapter iii holds, yet x_0 exceeds the right-hand side of (8) only a little. In this case, as class A weakens due to a relative decrease of n_A^A, it may happen that the inequality

$$n_B{}^B + n_{II}{}^B > \frac{a}{a_0 + a} N + \frac{c_0 - a}{a_0 + a} n_A{}^A \qquad (24)$$

will begin to hold, where a_0 refers to individuals of type I_B in classes II and B, while c_0 refers to individuals of type I_A in class A. We obtain an expression similar to equation (20) for the time when this occurs. In this case, individuals of type I_B gain control of the population, and the general behavior changes from A' to B.

The actual transition either from A' to A'' or from A' to B is described by equations of the type of equation (6) of chapter iii or of other types studied in chapter iii. Once a new kind of behavior is established, two things may happen. Either the principle of "controlling class heredity," in other words of small mobility, is part of the new behavior, or it is not. In the first case, again a "thinning out" of the controlling class will occur, and this will result in a change from the new behavior to the old one, these changes going on periodically, with a period of the order of a few hundred years. It must be emphasized that the control which a class exercises need not necessarily be political. It may be a cultural control. In general these two will likely be correlated.

If, after a change from behavior pattern A to behavior pattern B, the principle of class heredity is not adhered to, the behavior pattern B may last indefinitely. This, however, is not necessarily the case. We have seen that [cf. equations (12)]

$$n_A{}^A + n_{II}{}^A + n_B{}^A = \alpha_A N; \quad n_A{}^B + n_{II}{}^B + n_B{}^B = \alpha_B N. \qquad (25)$$

If $\alpha_A >> \alpha_B$, then eventually the number of individuals of type I_A will increase sufficiently for them to gain control of the population. In spite of a large α_A, and even due to it, control by individuals of type I_A cannot last indefinitely because, due to the principle of class heredity, the group of individuals of type I_A eventually divides itself into A' and A''. But a behavior pattern which does not adopt class heredity will *under these conditions* also not exist indefinitely.

We have studied here the case of an exponentially increasing population. The other extreme, that of a stationary population, can also be easily studied. In this case, N remains constant, and n_A, n_{II}, and n_B also remain constant, but $n_A{}^A$, $n_{II}{}^A$, etc., vary. One should also investigate the case of interaction of three or more groups of type I.

Periodical fluctuations of influence of two different groups may also occur in the absence of class heredity if we consider that a group associates into a class only upon reaching a certain size, and that the mortality in such a *closed* class is greater than in class II because of inbreeding or for other reasons. But until a class is formed, the

mortality of individuals I_A is the same as that of the other individuals of class II. Then such a class A will gradually become relatively smaller until it is overcome by another group. However, since class II produces individuals of type I at a rate proportional to α, then if α is sufficiently large, a group I_A will be regenerated, and keep on regenerating until it becomes so large that it forms an exclusive class, which results in an increase of mortality, etc. The discussion of the differences of mortalities in different social groups by P. Sorokin[2] is of interest in this connection.

II

The slow variations of the social structure studied in chapter v and based on economic interaction may be studied in a similar way. Consider the case described by equations (15) and (17) of chapter v. Originally we have N_1 individuals of type I forming class I, and $N_2 > N_1$ individuals of type II forming class II. Let us consider again that the progeny of parents of a given class continues to belong nominally to the same class, although it may be of a different type. Denoting by n_1' the number of individuals of type I in class I, by n_1'' the number of individuals of type II in class I, by n_2' the number of individuals of type I in class II, and by n_2'' the number of individuals of type II in class II, let us put:

$$n_1' + n_1'' = n_1; \qquad n_2' + n_2'' = n_2. \tag{26}$$

Denoting by β the rate of increase of the population, assumed the same for all types, and by α the fraction of individuals of type I born from any parents, we again have:

$$\frac{dn_1}{dt} = \beta n_1; \quad \frac{dn_2}{dt} = \beta n_2; \tag{27}$$

$$n_1 = n_{01}\, e^{\beta t}; \quad n_2 = n_{02}\, e^{\beta t}; \tag{28}$$

and

$$\frac{dn_1'}{dt} = \alpha\beta n_1; \quad \frac{dn_1''}{dt} = (1-\alpha)\beta n_1; \qquad \frac{dn_2'}{dt} = \alpha\beta n_2; \quad \frac{dn_2''}{dt} = (1-\alpha)\beta n_2. \tag{29}$$

Remembering that for $t = 0$, $n_1' = n_{01}$, $n_1'' = 0$; $n_2' = 0$; $n_2'' = n_{02}$, we find from equations (29) and (28)

$$n_1' = an_{01}(e^{\beta t} - 1) + n_{01}; \quad n_1'' = (1-a)n_{01}(e^{\beta t} - 1);$$
$$n_2' = an_{02}(e^{\beta t} - 1); \quad n_2'' = (1-a)n_{02}e^{\beta t} + an_{02}. \tag{30}$$

For simplicity we neglect here the effect of "vertical mobility" from class to class.[1]

According to the picture made here, these variations of the different n_k^i result in a variation of the quantity η, as defined in chapter v, for η is nothing else but n_1'/n_2. The quantities ε_1 and ε_2 are also affected by these changes, for when the whole class I of $n_1 = n_1' + n_1''$ individuals is not composed of individuals of type I, but of individuals of both types, we should substitute for ε_1 and ε_2 average values, such as

$$\bar{\varepsilon}_1 = \frac{n_1'\varepsilon_1 + n_1''\varepsilon_2}{n_1}; \quad \bar{\varepsilon}_2 = \frac{n_2'\varepsilon_1 + n_2''\varepsilon_2}{n_2}. \tag{31}$$

Remembering that n_2 is the same as N_2 of chapter v, putting

$$\eta = \frac{n_1'}{n_2}, \tag{32}$$

introducing equations (30) into (31) and (32), and then introducing the result into equations (19) and (18) of chapter v, we obtain differential equations for W_1 and W_2, which are, however, too cumbersome to be treated directly. In view of the schematizations already introduced, an exact solution of these equations would hardly be worth while. However, we may derive approximately some general properties of the solutions without actually solving the equations.

Let us again consider the ratio W_1/W_2. As we have seen, it originally increases, tending to a constant value given by equation (31) of chapter v. This value, however, remains constant only as long as η, ε_1, and ε_2 are constant. Since this is not the case, W_1/W_2 will continue to change. From expressions (28) and (30) it follows that n_1/n_2 remains constant, but that n_1'/n_1 decreases. Equation (31) shows that $\bar{\varepsilon}_1$ decreases from $\bar{\varepsilon}_1 = \varepsilon_1$ to $\bar{\varepsilon}_1 = \varepsilon_2 < \varepsilon_1$ while $\bar{\varepsilon}_2$ increases. The quantity η decreases because n_1'/n_2 decreases. This will result in a decrease of the numerator of expression (31) of chapter v and an increase of the denominator, hence in a decrease of W_1/W_2. We have specifically considered the situation in which originally $dW_1/dt > dW_2/dt$, in other words when [cf. equations (18) and (19) of chapter v]:

$$\varepsilon_1\eta + w_0 f(\eta) + b - 2\sqrt{bw_0 f(\eta)} > \varepsilon_2 - b + \sqrt{bw_0 f(\eta)}. \tag{33}$$

We shall investigate how inequality (33) is affected by a variation of η. In order to discuss this, we must make some assumptions about

$f(\eta)$. If $f(\eta)$ is of the form (7) of chapter v, then we may roughly approximate it for small values of η by

$$f = f_0 a \eta \tag{34}$$

in the range from $\eta = 0$ to $\eta = 1/a$, and by

$$f = f_0 \tag{35}$$

for $\eta > 1/a$. In other words, we substitute instead of the curve (Figure 1), two segments of straight lines. Since a variation of η will

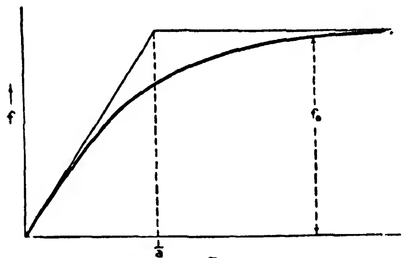

FIGURE 1

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affect $f(\eta)$ only for smaller values of η , we may substitute equation (34) into (33). We shall also neglect in the first approximation the changes of ε_1 and ε_2 . As a matter of fact, we shall considerably simplify our formulae without greatly affecting their generality by putting $\varepsilon_1 = \varepsilon_2 = 0$. Making the above substitutions into inequality (33) and substituting an equality sign for the inequality, we obtain the following equation for determination of η^*, the value of η for which inequality (33) breaks down:

$$w_0 f_0 a \eta^* + 2b = 3\sqrt{b w_0 f_0 a \eta^*} \, . \tag{36}$$

Squaring and solving for η^* we find two positive roots

$$\eta_1^* = \frac{b}{w_0 f_0 a} \quad \text{and} \quad \eta_2^* = \frac{4b}{w_0 f_0 a} . \tag{37}$$

Since equations (37) are obtained by substituting equation (34) into inequality (33), and since equation (34) holds only for $\eta < 1/a$, therefore the above argument, as well as its subsequent consequences, holds only if the larger root η_2^* is less than $1/a$, or if $4b/w_0 f_0 < 1$. Because of the approximation used, our results hold only within a re-

stricted range of the constants. The meaning of the two roots becomes clear upon inspection of inequality (33). It can be easily shown that for $\varepsilon_1 = \varepsilon_2 = 0$ inequality (33) holds only for $\eta > \eta_2^*$ and for $\eta < \eta_1^*$, while for $\eta_1^* < \eta < \eta_2^*$ the inequality (33) is reversed. Physically, the meaning of this situation can also be easily understood. A decrease of η means a decrease of proper supervision or organization of the work of individuals of type *II*, and a corresponding decrease in output. For $\eta_1^* < \eta < \eta_2^*$, this supervision becomes too small to enable class *I* to retain a sufficient amount of goods relative to class *II*. Still the work of class *II* produces enough goods to make both dW_1/dt and dW_2/dt positive. For $\eta > \eta_1^*$, both sides of inequality (33) are positive. When η decreases still further, approaching zero, the individuals of class *II* cannot produce enough and again $dW_1/dt > dW_2/dt$, but both are negative. The first class merely loses less than the second.

We thus see that since η decreases with time, following a period of "concentration" of accumulated wealth in class *I*, there is a period of "deconcentration." The beginning of the "decline" of class *I* is given by the equation

$$\eta = \eta_2^*, \tag{38}$$

in which we have to substitute for η_2^* the expression (37), for η the expression (32), and then introduce for n_1' and n_2 the expressions (28) and (30). This gives

$$\frac{an_{01}(e^{\beta t} - 1) + n_{01}}{n_{02}\, e^{\beta t}} = \frac{4b}{w_0 f_0 a}. \tag{39}$$

Solved with respect to t, this gives:

$$t = \frac{1}{\beta} \log \frac{(1 - a)\, n_{01} w_0 f_0 a}{4bn_{02} - an_{01} w_0 f_0 a}. \tag{40}$$

In order that t should be real we must have

$$4bn_{02} > an_{01} w_0 f_0 a\,, \tag{41}$$

which imposes an upper limit on n_{01}/n_{02}. The meaning of this becomes clear when we consider that with the assumption made here about $f(\eta)$, an increase of n_{01}/n_{02} above a certain limit leaves $f(\eta)$, and therefore dW_1/dt and dW_2/dt, practically unchanged. On the other hand, n_1' tends, as seen from equations (30) and (28), to an_1. Hence $\eta = n_1'/n_2$ tends to the value $an_1/n_2 = an_{01}/n_{02}$. If originally there has been a very large "reserve" of n_{01}/n_{02}, then this final value of η may still be sufficient under these conditions to keep $dW_1/dt > dW_2/dt$.

In reality such a case is very unlikely to occur. It would actually mean a very large "initial reserve" of executive and supervisory personnel. In any case, the ratio W_1/W_2 would decrease due to the decrease of ε_1 and increase of ε_2, a fact which we neglected in our approximation.

Moreover, we must consider that the number n_2' of individuals of type I in class II is gradually increasing. When the ratio $\eta' = n_2'/n_2$ becomes sufficiently large, a new class of I' of "supervisors" will be formed within class II, and I' will begin to compete with class I. Depending on the choice of the constants, this may happen even before class I "declines" appreciably spontaneously.

The rise of class I' and its competition with I will begin when η' satisfies the equation

$$\begin{aligned} &\varepsilon_1\eta + w_0 f(\eta) + b - 2\sqrt{b w_0 f(\eta)} \\ &\quad = \varepsilon_1\eta' + w_0 f(\eta') + b - 2\sqrt{b w_0 f(\eta')}, \end{aligned} \tag{42}$$

which requires that $\eta = \eta'$ and in which we have to put [because of equations (28) and (30)]

$$\eta = \frac{n_1'}{n_2} = \frac{\alpha n_{01}(e^{\beta t} - 1) + n_{01}}{n_{02}e^{\beta t}};$$

$$\eta' = \frac{n_2'}{n_2} = \frac{\alpha(e^{\beta t} - 1)}{e^{\beta t}}.$$

This gives

$$t = \frac{1}{\beta}\log\frac{(1-\alpha)n_{01} + \alpha n_{02}}{\alpha(n_{02} - n_{01})}. \tag{43}$$

We may consider a more complicated case, in which, because of the difference of the amounts of wealth already accumulated, the competitor, in order to succeed, must start with more stringent conditions than equation (42). For instance, we may require that the right-hand side of (42) exceed the left-hand side by an amount which is a function of the difference of accumulated W_1 and W_2. This will give for t, instead of equation (43), an expression which will involve such constants as ε_1, ε_2, f_0, a and b.

The interesting result of this study is that it shows us a simple picture of the "birth" of a "wealthy" class, with an ensuing initial "concentration" of "wealth," followed by a "decline" and finally succeeded by a class composed of different individuals. While a differentiation according to accumulated wealth remains, yet the class distinction shifts with time from one group of individuals to another. When developed much farther, such a study may perhaps be connected

with P. Sorokin's interesting study on the "life-span" of different social organizations.[4] Comparisons of calculated life-span [e.g. equation (40)] with the observed ones may give us the values of some of the constants involved.

The most interesting further developments, however, would be in connection with the preceding study of the life-span and interaction of classes which exercise a control due to their initiative, regardless of the amount of wealth. The existence of a "wealthy" class is, after all, possible only when the majority of the individuals in class *II* regulate their behavior according to the principles involving the sanctity of private property. Otherwise, they may take away the accumulated wealth of class *I* by sheer force, due to their numerical prevalence. Thus, the existence of such a class *I* presupposes either an influence of the individuals of this class on those of type *II*, or the existence of another "influencing" group which determines the behavior of individuals of type *II*. In the latter case, any changes of the interrelations between the influencing group and the rest of the population will have strong repercussions on the relation of class *I* to class *II* and on W_1/W_2. Such relations should form the object of further studies.

III

We shall now discuss the interaction of three classes, an autocratic, an organizing, and a passive one, as studied in chapter vi.

To connect the expressions derived on page 57 with the problem of the variation of the socio-economic structure with time, we must express the quantities A, B, C, a, and b in terms of N_1, N_2, and N_3. It will be rather natural to assume that a and b are proportional to N or N_3, since $N_1 << N_3$ and $N_2 << N_3$. The demand will approximately double if the number of consumers doubles, *ceteris paribus*. Concerning A, B, and C, we may consider that the greater class *II*, in other words, the more people there are capable of supervising the work, the less will be q. With an increase of N_2, the number of inventors will also increase, facilitating production and therefore reducing q, since inventors come from class *II*. Assuming, *only as an illustration*, that

$$A \propto \frac{1}{N_2}; \quad B \propto \frac{1}{N_2}; \quad C \propto \frac{1}{N_2};$$
$$a \propto N_3; \quad b \propto N_3, \tag{44}$$

let us first consider the simplest possible case in which the total population increases but the ratio N_2/N_3 remains constant. Then we see

from equation (15) of chapter vi that ξ_m will first increase, then tend asymptotically to a constant value. For very large N_3, u will grow approximately as N_3, as do p and q. Hence we have

$$\xi u \propto N_3; \quad q \propto N_3, \quad u \propto N_3; \quad p \propto N_3 .$$

The ratio $\xi u/(pu - q - \xi u)$ of the amounts retained by classes I and II will therefore vary, with larger N's, as $1/N_3 \propto 1/N$. With increasing population the relative rate of accumulation of wealth by class I will be decreased as compared with that of class II. The relative wealth of the two classes will also decrease. It must be noted that the ratio $\xi u/(pu - q - \xi u)$ may initially be either greater or less than $1/2$, depending on the values of the constants. Thus the controlling class need not necessarily be the wealthiest. In any case, however, under the assumptions made here, the controlling class will gradually become *relatively* poorer as compared with class II.

We have assumed that N_2/N_3 remains constant as N increases. This means an unrestricted social mobility between class II and class III. This is probably what actually happens for the class of people characterized by some special abilities of technical and scientific nature. If we assume that between class I and class III the social mobility is small, then, as has been shown in chapter iii, after a lapse of time inequality (2) of chapter vi will cease to hold and the control will pass to class II. Expression (20) has been derived for the moment t^* when this will happen. For $t > t^*$ the relations discussed in this section do not hold, for class II does not need to give anything to class I. Hence, if at the moment t^* class I is still relatively wealthy, then at $t = t^*$ there will be a sudden drop in the relative wealth of classes I and II. On the other hand, at $t = t^*$ class I may already be sufficiently less wealthy than class II, in which case the discontinuity will be either less pronounced or altogether absent.

In deriving expression (20) we assumed the coefficients a_0, c_0, and a in equations (17) to be constant. Actually these coefficients will increase with increasing wealth of the corresponding classes because they increase with the amount of technical facilities available for communication. Thus by making a_1, a_2, and a_3 functions of time, through their functional relation to the total amounts of wealth available, we shall have a more complex equation for t^*. This is an interesting problem which requires a special investigation.

IV

Now we shall discuss the long range variations of social structure for the case discussed on page 59, in chapter vi, which leads to

equations (25) and (28) of that chapter.

We have

$$a_1' >> b_1' , \quad a_2' << b_2' , \quad a_1' >> a_2' , \qquad b_1' << b_2' . \tag{45}$$

For a_1 , a_2 , b_1 , and b_2 we have

$$a_1 = n_I{}^I a_1' + n_I{}^{II} a_2' , \quad a_2 = n_{II}{}^I a_1' + n_{II}{}^{II} a_2' ,$$
$$b_1 = n_I{}^{II} b_1' + n_I{}^{II} b_2' , \quad b_2 = n_{II}{}^I b_1' + n_{II}{}^{II} b_2' . \tag{46}$$

While a_1' , b_1' , a_2' , b_2' are constants, the values of a_1 , b_1 , a_2 , and b_2 vary with respect to time because of variations of $n_I{}^I$, $n_I{}^{II}$, etc. With the foregoing assumptions, $n_I{}^I / n_I{}^{II}$ decreases with time, according to equations developed previously, while $n_{II}{}^I / n_{II}{}^{II}$ remains constant. On the other hand, all $n_i{}^k$ increase. Thus a_1 , a_2 , b_1 , and b_2 will all increase. But, because of equation (28) of chapter vi, a_1 and b_1 will increase more slowly than a_2 and b_2 , so that a_1/a_2 and b_1/b_2 decrease. Hence, according to equations (26) and (28) of chapter vi, x_1/x_2 and y_1/y_2 will decrease. In other words, while class *I* receives the lesser fraction of goods from class *II* , at the same time the *relative* amount of legislative privileges of class *II* as compared with class *I* increases. In Figure 2 the variations of x_1/x_2 , as given by equation (27) of

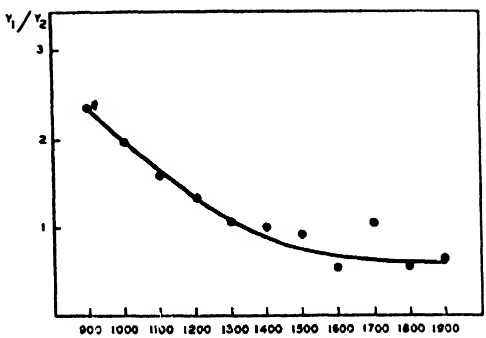

FIGURE 2

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chapter vi, are shown for the following choice of constants: $a_1' = 10a_2'$, $b_2/b_1 = 1{\cdot}06$. The rates of increase of the population for all classes are assumed the same, with $\beta = 0{\cdot}062$ years^{-1}. The percentage α_1 of offspring of type *I* is taken as $0{\cdot}036$.

For comparison, points plotted in Figure 2 are taken from data

compiled by P. Sorokin.[2] In collaboration with a number of other economists and sociologists, Sorokin has made tentative estimates of the economic conditions for different social classes in different countries. The estimates are based on a relative scale, ranging from 1 to 10. As Sorokin points out, the individual curves do not show any definite trends. If we take the *ratio*, for instance, of the values for the nobility and the bourgeoisie in France, we find a definite downward trend, as seen from Figure 2.

It may be legitimately questioned as to whether this ratio can be identified with our x_1/x_2. Such an identification is justified only as a rough approximation. The relative economic conditions of the two classes at a given moment are functions not only of x_1/x_2 at that moment, but also of the values of x_1/x_2 at previous times, since wealth can be accumulated. However, we may roughly compare the relative economic condition of two classes with x_1/x_2. The bourgeois class was chosen as representing class II, which is also only approximately true. No definite conclusions should be drawn from Figure 2, which is given merely as an *illustration* of how some theoretical conclusions of the type discussed above might be compared to sociological data, *if proper data were available.*

One could in principle apply similar considerations to the variations of civil laws with time, which may characterize the different rights, privileges, and obligations of different classes and may be identified with y_1 and y_2. Quantitative data of that type are very difficult to obtain. It must be noted, however, that P. Sorokin[3] has made an interesting attempt to determine the quantitative indices for the variations of criminal laws. An extension of Sorokin's method to civil laws would be very desirable.

Using expression (22) of chapter vi without any restriction upon the coefficients, we obtain similar, though more complicated, relations.

At this point it must be very strongly emphasized that the mathematical theory of legislative change suggested above does not constitute an "economic interpretation" of social phenomena and does not commit us to such an interpretation. Neither does it commit us to any other special sociological doctrine. At first glance one might think that the shift of legislative tendencies, expressed by y_1/y_2, is the result of the shift of the ratio x_1/x_2, which is an economic quantity. The true cause of the variation, however, is the "thinning out" of class I, which results in a loss of a number of capable legislators in it. The parallel variation in x_1/x_2 and y_1/y_2 is a consequence of this "thinning out" plus the special assumption made about the satisfaction functions. With somewhat more general assumptions about these functions, we shall in general find $x_1/x_2 \neq y_1/y_2$, though the two

ratios may exhibit a certain parallelism in their variations.

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CHAPTER XV

VARIATIONS OF THE PROFILE OF A SOCIAL GROUP WITH TIME

In this chapter we shall discuss how the relative numbers of individuals of different types in the whole society can vary with time. This may be due to different birth and death rates for different types. The following discussion is intended as an illustration only. The simple mechanism discussed here is not likely to occur actually.

We shall again consider three types of people: one active, characterized by a behavior A, another active, characterized by a behavior B, and a passive one. We shall change our notations, however, and denote the number of individuals of each type by x, y, and z, respectively. We shall first restrict ourselves to a strictly selective mating, so that only persons of the same type intermarry.

The total birth rates for different types of matings will in general be different. Of all those total birth rates in each type of mating there will be given fractions of offspring born of each type, these fractions also being different for different types of matings. The determination of these fractions is a problem of genetics, and shall not be considered here.

The total rate of increase of x with respect to t will be the sum of terms proportional to x, y and z, respectively. To that sum must be added a negative term proportional to x, which expresses the death rate of individuals of the first type. Similar considerations apply to the rate of increase of y and z. We thus find

$$\begin{aligned} \frac{dx}{dt} &= a_{11}x + a_{12}y + a_{13}z, \\ \frac{dy}{dt} &= a_{21}x + a_{22}y + a_{23}z, \\ \frac{dz}{dt} &= a_{31}x + a_{32}y + a_{33}z. \end{aligned} \tag{1}$$

There is a theoretical possibility that a_{11}, a_{22}, a_{33} may be negative. Thus a_{11} will be negative if the death rate of the individuals of the first type is greater than its birth rate from type x. Similarly, this may happen for a_{22} and a_{33}. If we consider a rapidly growing

population in which death rates are much smaller than birth rates, such an occurrence will be biologically extremely unlikely. The birth rate for a given type will most likely be much higher from matings between individuals of the same type than from any others. Thus the more probable situation is that a_{11}, a_{22} and a_{33} are not only positive, but are also much larger than the other a_{ik}'s.

The general solution of the system (1) is of the form

$$\begin{aligned} &G_{11}e^{\lambda_1 t} + G_{12}e^{\lambda_2 t} + G_{13}e^{\lambda_3 t}, \\ &G_{21}e^{\lambda_1 t} + G_{22}e^{\lambda_2 t} + G_{23}e^{\lambda_3 t}, \\ &G_{31}e^{\lambda_1 t} + G_{32}e^{\lambda_2 t} + G_{33}e^{\lambda_3 t}, \end{aligned} \tag{2}$$

where the G's are constants and λ_1, λ_2 and λ_3 are the roots of the characteristic equation

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{vmatrix} = 0 . \tag{3}$$

Equation (3) is of the form

$$-\lambda^3 + A\lambda^2 + B\lambda + C = 0 , \tag{4}$$

with

$$\begin{aligned} A &= a_{11} + a_{22} + a_{33}; \\ B &= a_{21}a_{12} + a_{32}a_{23} + a_{31}a_{13} - a_{11}a_{33} - a_{22}a_{33} - a_{11}a_{22}; \\ C &= a_{11}a_{22}a_{33} + a_{31}a_{12}a_{23} + a_{21}a_{32}a_{13} - a_{11}a_{32}a_{23} - a_{21}a_{12}a_{33} \\ &\quad - a_{31}a_{22}a_{13} . \end{aligned} \tag{5}$$

From what was said above we may expect that in general (though not necessarily always) the coefficients a_{11}, a_{22} and a_{33} will be considerably larger than the remaining coefficients. Under these conditions

$$A > 0; \quad B < 0; \quad C > 0 . \tag{6}$$

An elementary graphical examination of equation (4) with conditions (6) shows that it has one real root and two conjugate complex roots, the real part of the latter roots being positive and less than the value of the real root. This is also readily ascertained by taking, for instance, as a numerical example $a_{11} = a_{22} = a_{33} = 2$, all other a_{ik}'s being equal to 1. (Only relative values of the coefficients are important.) Thus we have

$$\lambda_1 = u + iv , \quad \lambda_2 = u - iv, \quad \lambda_3 = w , \tag{7}$$

where u, v and w are real numbers and

$$u < w . \tag{8}$$

The first two roots λ_1 and λ_2 give an oscillation with exponentially increasing amplitude, of the form

$$e^{ut} \sin vt \,, \tag{9}$$

while λ_3 gives a term of the form

$$e^{ut} \,.$$

The zero point of the oscillations thus increases exponentially. Because of inequality (8) the amplitude of the oscillatory term increases less rapidly than the zero point. Therefore, if proper physically meaningful initial conditions are chosen, the values of x, y and z will oscillate around exponentially increasing positive values but will remain positive. In this case the relative sizes of the two active groups will fluctuate even in the absence of any changes produced by "class heredity." The conditions (9) and (8) of chapter iii will also alternate and we shall have oscillations between two types of behavior for the whole group. It must be remembered, however, that these changes will not necessarily be periodical, because our solutions contain a non-periodical term, and the inequalities (8) and (9) of chapter iii involve all terms. With plausible values of the constant a_{ik} we find that the period $2\pi/v$ in (9) is of the order of 10^2–10^3 years.

If we consider a more general case—where there is a certain percentage of intermarriages between different types—then we must add to the right sides of equations (1) terms expressing the birth rates from such mixed intermarriages. The birth rate of individuals of type *I* from intermarriages between type *I* and type *II* will be proportional to the number f_{xy} of such intermarriages. The number f_{xy} is itself determined by the statistical distribution of intermarriages of the kind considered, and is *in general* a function of x, y and z. Similar considerations hold for f_{xz} and f_{yz}.

If the functions f_{xy}, f_{xz}, and f_{yz} are analytic everywhere, they can be developed around $x = y = z = 0$ into series not containing any linear terms, for f_{xy} must obviously be zero when either x or y is zero. Therefore instead of equation (1) we shall have:

$$\begin{aligned}
\frac{dx}{dt} &= a_{11}x + a_{12}y + a_{13}z + \sum_{i,k,l} g_{ikl}x^i y^k z^l \,, \\
\frac{dy}{dt} &= a_{21}x + a_{22}y + a_{23}z + \sum_{i,k,l} p_{ikl}x^i y^k z^l \,, \\
\frac{dz}{dt} &= a_{31}x + a_{32}y + a_{33}z + \sum_{i,k,l} r_{ikl}x^i y^k z^l \,,
\end{aligned} \tag{10}$$

where the sums on the right side do not contain any linear terms and the a_{ik}'s are all positive.

The general solution of the system (10) is of the form[1]

$$\begin{aligned} x &= G_{11}e^{\lambda_1 t} + G_{12}e^{\lambda_2 t} + G_{13}e^{\lambda_3 t} + \cdots + G_{1kl}e^{k\lambda_l t} + \cdots, \\ y &= G_{21}e^{\lambda_1 t} + G_{22}e^{\lambda_2 t} + G_{23}e^{\lambda_3 t} + \cdots + G_{2kl}e^{k\lambda_l t} + \cdots, \\ z &= G_{31}e^{\lambda_1 t} + G_{32}e^{\lambda_2 t} + G_{33}e^{\lambda_3 t} + \cdots + G_{3kl}e^{k\lambda_l t} + \cdots, \end{aligned} \tag{11}$$

where again the G's are constants and the λ's are the roots of equation (4). We still have periodic fluctuations of x, y, and z around monotonically increasing points; those fluctuations, however, are not simple harmonics, but are represented by Fourier's series of corresponding fundamental frequency v.

If f_{xy}, f_{yz}, and f_{xz} are not everywhere analytic, the foregoing argument does not hold, and we cannot tell whether periodicities exist or not. A special investigation is necessary. It may be remarked that some rather simple and natural assumptions about the distribution of matings of different types may lead to expressions for f_{xy}, f_{xz}, and f_{yz} which are not analytic at $x = y = z = 0$.

It is readily seen that in the case with only two types of individuals, which leads to a system of two equations, periodical solutions are impossible if the population has to increase. The characteristic equation is in that case a quadratic one, with two roots λ_1 and λ_2. Either both λ's are real, or they are conjugate complex. In the latter case we have oscillations with variable amplitude around a fixed point. In fact, in the absence of constant terms in the expansions on the right sides of (10), that fixed point is zero, leading to negative values for the variables, which is physically absurd.

On the other hand, with more than three variables, even more complicated cases of periodical fluctuations are likely to occur.

Much more complicated non-linear equations are obtained if we consider different possible types of selective matings between different classes. This opens a wide and as yet untouched field for further investigations.

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CHAPTER XVI

ANOTHER CASE OF VARIATIONS IN THE STRUCTURE OF A SOCIAL GROUP WITH TIME

In the case considered in chapter xv we found that the periodic oscillations may occur around values which grow exponentially with time. In the present chapter we shall consider a somewhat different situation, where the size of the social group approaches a limiting value.

It is known[1] that in many cases the growth of human population of N individuals is well represented by the following differential equation:

$$\frac{dN}{dt} = a'N - b'N^2, \quad a' > 0, \quad b' > 0 . \tag{1}$$

The natural biological interpretation of equation (1) is to assume a constant birth rate a per individual and a death rate which is linearly increasing with the density of population N/S, where S denotes the area occupied by the population. Putting for the death rate $\left(b + \frac{c'}{S} N\right) N$, and putting $c'/S = c$, we obtain:

$$\frac{dN}{dt} = aN - (b + cN) N ,$$

which can be written in the form of equation (1):

$$\frac{dN}{dt} = (a - b) N - cN^2. \tag{2}$$

Let us consider that the population consists of two types of individuals, their corresponding numbers being N_1 and N_2, so that

$$N_1 + N_2 = N . \tag{3}$$

We may in general consider the following system of equations as representing the variation of N_1 and N_2:

$$\begin{aligned} \frac{dN_1}{dt} &= a_1'N_1 + a_1''N_2 - [b_1 + c_1(N_1 + N_2)] N_1 , \\ \frac{dN_2}{dt} &= a_2'N_1 + a_2''N_2 - [b_2 + c_2(N_1 + N_2)] N_2 . \end{aligned} \tag{4}$$

The first two terms on the right side of each equation represent the contributions of the two groups to the birth rates, while the last terms represent death rates, which for both types are increasing with the *total* density of population, but they increase at different rates. It would be of interest to study the system (4), or possibly more general systems. The non-linearity of the system (4) makes this a rather difficult task. We shall therefore consider here a limiting case which may be of sociological importance.

Let N_1 denote the number of active individuals in the population, while N_2 denotes the number of passives. Furthermore, as will usually be the case, let $N_1 << N_2$, so that we may put

$$N_1 + N_2 = N_2 = N . \tag{5}$$

The second equation (4) then represents essentially the growth of the total population. The first term of the right side may be omitted, and we obtain from it equation (2) by putting $a_2'' = a$, $b_2 = b$, $c_2 = c$. As for the first equation of (4), we shall first consider the case where, due to the smallness of N_1, the contribution of the active group to the birth rate of the actives is negligible as compared with the contribution of the passives. In other words, we neglect $a_1'N_1$. Later we shall drop this restriction. Of the aN individuals born in the total population, let the fraction $\alpha < 1$ be active individuals. Then $a_1'' = \alpha a$ and the first equation (4) becomes

$$\frac{dN_1}{dt} = \alpha a N - (b_1 + c_1 N) N_1 . \tag{6}$$

Of particular interest is the case $c_1 < c_2$, in other words, when the increase in density of population affects the death rate of the actives less than that of the passives. An increase in density of population means in general more difficult conditions of living and a greater struggle for existence. The active individual is more likely to meet those increased demands, and thus the assumption $c_1 < c$ is natural. As may be expected, and as shall be seen later, this results in an increase of the ratio N_1/N with increasing N.

The integral of equation (2) is

$$N = \frac{A}{1 + e^{-Bt}} \tag{7}$$

where

$$A = \frac{a - b}{c}; \quad B = a - b; \tag{8}$$

and the origin of time is chosen so that at $t = 0$, $N = A/2$. The total population tends to $N_\infty = A$.

The integral of equation (6) is

$$N_1 = e^{-\int (b_1 + c_1 N) dt} \{C + \alpha a \int N e^{\int (b_1 + c_1 N) dt} \, dt\}, \tag{9}$$

where C is a constant of integration.

From equations (7) and (9) we obtain

$$e^{-\int (b_1 + c_1 N) dt} = e^{-b_1 t} \left(\frac{e^{-Bt}}{1 + e^{-Bt}} \right)^{c_1/c}, \tag{10}$$

while

$$\int N e^{\int (b_1 + c_1 N) dt} \, dt = A \int e^{\left(b_1 + \frac{Bc_1}{c}\right)t} (1 + e^{-Bt})^{c_1/c - 1} \, dt. \tag{11}$$

Putting

$$e^{-Bt} = x; \quad \frac{c_1}{c} - 1 = p; \quad - \frac{b_1 c + Bc_1}{Bc} = k, \tag{12}$$

and using equations (8), we find for the integral (11) the expression:

$$-\frac{1}{c} \int x^{k-1} (1 + x)^p \, dx. \tag{13}$$

For arbitrary values of k and p, convenient closed expressions for (13) are not available. To obtain an idea of the variation of N_1 with either t or N, we shall study a few special cases obtained by using plausible values of the constants.

Case 1. $a = 2b$; $b_1 = b$; $c_1 = 0$. Then

$$A = \frac{b}{c}; \quad B = b = b_1; \quad k = -1; \quad p = -1. \tag{14}$$

Expression (13) now becomes[2]

$$-\frac{1}{c} \int \frac{dx}{x^2 (1 + x)} = \frac{1}{c} \frac{1 + x}{x} - \frac{1}{c} \log \frac{1 + x}{x}. \tag{15}$$

Equations (9), (10), (11), (13) and (15) now give

$$N_1 = e^{-bt} \left\{ C + \frac{\alpha a}{c} \frac{1 + e^{-bt}}{e^{-bt}} - \frac{\alpha a}{c} \log \frac{1 + e^{-bt}}{e^{-bt}} \right\}. \tag{16}$$

For $t = \infty$ we have $N_1 = \alpha a / c$. This agrees also with equation (6). The latter gives as a limiting stationary value

$$N_{1\infty}=\frac{aaN_\infty}{b_1+c_1N_\infty}. \tag{17}$$

Because of $N_\infty = A$ and because of (14), this reduces to aa/c.

From equation (7) we have

$$1+e^{-Bt}=\frac{A}{N}; \quad e^{-Bt}=\frac{A-N}{N}. \tag{18}$$

Substituting this into equation (16), we find:

$$N_1=\frac{A-N}{N}\left\{C+\frac{aa}{c}\frac{A}{A-N}-\frac{aa}{c}\log\frac{A}{A-N}\right\}. \tag{19}$$

If for $N=0$, $N_1=0$, we must have

$$C=-\frac{aa}{c}, \tag{20}$$

thus obtaining

$$N_1=\frac{aa}{c}\left(1+\frac{A-N}{N}\log\frac{A-N}{A}\right). \tag{21}$$

Since $y \log y$ tends to zero as y tends to 0, therefore $N_1 = aa/c$ for $N=A$.

Expanding $\log\dfrac{A-N}{A}=\log\left(1-\dfrac{N}{A}\right)$, we find

$$N_1=\frac{aa}{c}\left[1+\left(\frac{A}{N}-1\right)\left(-\frac{N}{A}-\frac{N^2}{2A^2}-\frac{N^3}{3A^3}-\cdots\right)\right]=$$
$$\frac{aa}{c}\left[\frac{1}{2}\frac{N}{A}+\left(\frac{1}{2}-\frac{1}{3}\right)\frac{N^2}{A^2}+\cdots\right]=\frac{aa}{c}\sum_{i=1}^{i=\infty}\frac{(N/A)^i}{i(i+1)}. \tag{22}$$

Thus for $N=0$, $N_1=0$. The expansion having only positive terms, the first and second derivatives are positive for $0 < N < A$. The (N_1, N) curve is thus convex downward.

For

$$N=0,\ \frac{dN_1}{dN}=\frac{aa}{2Ac}=\frac{aa}{2b}. \tag{23}$$

We have

$$\frac{dN_1}{dN}=\frac{aa}{c}\frac{1}{N}\left(\frac{A}{N}\log\frac{A}{A-N}-1\right), \tag{24}$$

and this shows that for

$$N = A, \quad \frac{dN}{dN_1} = \infty. \tag{25}$$

Case 2. $a = 2b$; $b_1 = \frac{1}{2} b$; $c_1 = \frac{1}{2} c$. Then

$$A = \frac{b}{c}; \quad B = b = 2b_1; \quad k = -1; \quad p = -\tfrac{1}{2}. \tag{26}$$

Expression (13) now becomes[2]

$$-\frac{1}{c}\int \frac{dx}{x^2\sqrt{1+x}} = \frac{1}{c}\frac{\sqrt{1+x}}{x} + \frac{1}{2c}\log\frac{\sqrt{1+x}-1}{\sqrt{1+x}+1}. \tag{27}$$

Equations (9) to (13) now give

$$N_1 = e^{-(1/2)bt}\left(\frac{e^{-bt}}{1+e^{-bt}}\right)^{1/2}\left\{C + \frac{\alpha a}{c}\frac{\sqrt{1+e^{-bt}}}{e^{-bt}} + \frac{\alpha a}{2c}\log\frac{\sqrt{1+e^{-bt}}-1}{\sqrt{1+e^{-bt}}+1}\right\}. \tag{28}$$

For $t = \infty$, $N_1 = \alpha a/c$, which again checks with equation (6) for this case.

Again using equations (18) we find:

$$N_1 = C\frac{A-N}{\sqrt{AN}} + \frac{\alpha a}{c} + \frac{\alpha a}{2c}\frac{A-N}{\sqrt{AN}}\log\frac{\sqrt{AN}-N}{\sqrt{AN}+N}. \tag{29}$$

Expanding

$$\log\frac{\sqrt{AN}-N}{\sqrt{AN}+N} = -\log\frac{1+\sqrt{N/A}}{1-\sqrt{N/A}},$$

we find

$$N_1 = C\frac{A-N}{\sqrt{AN}} + \frac{\alpha a}{c}\left[1 - \left(\sqrt{A/N} - \sqrt{N/A}\right)\left(\sqrt{N/A} + \frac{1}{3}\sqrt{N^3/A^3} + \frac{1}{5}\sqrt{N^5/A^5} + \dots\right)\right] = C\frac{A-N}{\sqrt{AN}} + \frac{\alpha a}{c}\sum_{i=1}^{i=\infty}\frac{2(N/A)^i}{4i^2-1}. \tag{30}$$

Hence, in order to have $N_1 = 0$ for $N = 0$, we must have $C = 0$ and equation (28) becomes

$$N_1 = \frac{\alpha a}{c}\left(1 + \frac{1}{2}\frac{A-N}{\sqrt{AN}}\log\frac{\sqrt{AN}-N}{\sqrt{AN}+N}\right). \tag{31}$$

Expansion (30) consists again of only positive terms, hence for $0 < N < A$, $dN_1/dN > 0$ and $d^2N_1/dN^2 > 0$ and the curve is again convex downward.

For

$$N = 0, \quad \frac{dN_1}{dN} = \frac{2\alpha a}{3Ac} = \frac{2\alpha a}{3b}. \tag{32}$$

Case 3. $a = 2b$; $b_1 = b$; $c_1 = \frac{1}{2}c$. Then

$$A = \frac{b}{c}; \quad B = b = b_1; \quad k = -\frac{3}{2}; \quad p = -\frac{1}{2}. \tag{33}$$

Expression (13) now becomes[2]

$$-\frac{1}{c}\int \frac{dx}{x^{5/2}\sqrt{1+x}} = \frac{2}{3c}\frac{\sqrt{1+x}}{x^{3/2}} - \frac{4}{3c}\frac{\sqrt{1+x}}{\sqrt{x}}. \tag{34}$$

In a similar way as before we now obtain, putting $C = 4\alpha a/3c$,

$$N_1 = \frac{2\alpha a}{3c}\left[1 + 2\frac{A-N}{N}\left(\sqrt{\frac{A-N}{N}} - 1\right)\right]. \tag{35}$$

The expansion of the expression in brackets is

$$1 + \left(\frac{2A}{N} - 2\right)\left(-\frac{N}{2A} - \frac{1\cdot 1}{2\cdot 4}\frac{N^2}{A^2} - \frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}\frac{N^3}{A^3} - \cdots\right.$$

$$= \left(1 - \frac{1\cdot 1}{4}\right)\frac{N}{A} + \left(\frac{1\cdot 1}{4} - \frac{1\cdot 1\cdot 3}{4\cdot 6}\right)\frac{N^2}{A^2} + \left(\frac{1\cdot 1\cdot 3}{4\cdot 6} - \frac{1\cdot 1\cdot 3\cdot 5}{4\cdot 6\cdot 8}\right)\frac{N^3}{A^3}\cdots \tag{36}$$

$$= \sum_{i=0}^{i=\infty} \frac{1\cdot 1\cdot 3\cdots(2i-3)}{4\cdot 6\cdots 2i}\,\frac{3}{2i+2}\,(N/A)^i.$$

Hence, with the above choice of C, $N_1 = 0$ for $N = 0$ and again $dN_1/dN > 0$ and $d^2N_1/dN^2 > 0$. That $N_1 = 2\alpha a/c$ for $N = A$ is seen directly from equation (35).

We now shall drop the restriction made in obtaining equation (6) from the first equation (4), namely, the neglecting of $a_1'N_1$ in equations (4). Instead of expression (6) we then have

$$\frac{dN_1}{dt} = \alpha aN + a_1'N_1 - (b_1' + c_1N)\,N_1. \tag{37}$$

Equation (37) reduces to (6) if we put

$$b_1 = b_1' - a_1'. \tag{38}$$

The only difference between the new case and the previously studied ones is that now b_1 may be zero or negative, whereas hitherto it has always been positive. To see whether this leads to any different results, we shall investigate equation (6) for the case

$$b_1 = 0; \quad c_1 = \tfrac{1}{2} c; \quad a = 2b; \quad k = -\tfrac{1}{2}; \quad p = -\tfrac{1}{2}. \tag{39}$$

Expression (13) now becomes

$$-\frac{1}{c}\int \frac{dx}{x^{3/2}\sqrt{1+x}} = \frac{2}{c}\sqrt{\frac{1+x}{x}}, \tag{40}$$

and we finally obtain

$$N_1 = \sqrt{\frac{A-N}{A}}\left\{C + \frac{2\alpha a}{c}\sqrt{\frac{A}{A-N}}\right\}. \tag{41}$$

For $N = 0$

$$N_1 = C + \frac{2\alpha a}{c}.$$

Hence, if for $N = 0$, $N_1 = 0$, we must have $C = -2\alpha a/c$, and equation (41) becomes

$$N_1 = \frac{2\alpha a}{c}\left(1 - \sqrt{\frac{A-N}{A}}\right). \tag{42}$$

For $N = A$, $N_1 = 2\alpha a/c$.

We also have

$$\frac{dN_1}{dN} = \frac{\alpha a}{c\sqrt{A(A-N)}} > 0 \qquad \text{for} \quad 0 < N < A.$$

$$\frac{d^2N_1}{dN^2} = \frac{A\alpha a}{2c[A(A-N)]^{3/2}} > 0 \quad \text{for} \quad 0 < N < A.$$

We also see that $dN_1/dN = \infty$ for $N = A$.

Although we have thus far studied only special cases, we have considered a sufficiently wide range for the constants so that we may have a general idea of the variation of N_1 with N. The essential feature is that N_1 first increases linearly with N_0. However, for values of N in the neighborhood of $A/2$, N_1 increases much more rapidly, and near $N = A$ the rate of increase becomes still more rapid.

A variation of the ratio N_1/N will occur also in an indefinitely increasing population, with differential death ratio. If aN is the birth rate and bN the death rate for the whole population, we have

$$\frac{dN}{dt} = (a - b)\,N; \quad N = N_0 e^{(a-b)t}; \quad a > b. \tag{43}$$

Assuming again $N_1 << N$, let us assume

$$\frac{dN_1}{dt} = \alpha a N - b_1 N_1. \tag{44}$$

Again denoting by C a constant of integration, we have from expressions (43) and (44)

$$N_1 = e^{-b_1 t}\,\{C + \alpha a N_0 \int e^{(a-b)t + b_1 t}\,dt\} = C\,e^{-b_1 t} + \frac{\alpha a N_0}{a - b + b_1}\,e^{(a-b)t}. \tag{45}$$

With increasing t the ratio of N_1/N tends, because of expressions (43), to

$$\left(\frac{N_1}{N}\right)_\infty = \frac{\alpha a}{a - b + b_1}. \tag{46}$$

If $b_1 < b$, that ratio is greater than α, the ratio of birth rates of actives and passives.

A different mechanism, which will produce an increase in the ratio of active to passive individuals with increasing N, is suggested by the following considerations.

The active individual is more likely to possess some relatively less frequent characteristics than a passive one. It is likely, contrary to the assumption made in chapter xiv, that the fraction of individuals born to active parents is much larger than the fraction born to either passive parents or to one active and one passive parent.

In the early stages of history of a country, when the population density is very small and means of communications are primitive, the density of active individuals will be almost zero. The chances of one active individual meeting another one would be rather small, and a large percentage of intermarriages will be between an active and a passive individual. With increasing population density the relative incidence of active-active intermarriages will increase, and this will increase the relative birth rate of actives. It is of interest to formulate the above hypothesis in a mathematical form and study its quantitative consequences.

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CHAPTER XVII

SOME CONSEQUENCES AND POSSIBLE APPLICATIONS OF THE FOREGOING. INTERACTION OF NATIONS

The investigations of the last chapter make it plausible that with the growth of the total population of a nation, the ratio of the total number of various active individuals to the total population increases. We should therefore expect, in general, that for the two different nations this ratio N_1/N will be larger for the nation with a larger population density δ. Since the ratio N_1/N depends not only on the dynamics of the population growth, but also on the initial conditions, therefore we cannot expect N_1/N to be merely a function of δ. It will also be a function of those initial conditions and of time. Further mathematical studies may eventually reveal the nature of the functional relations to be expected.

To illustrate what such studies may give us, let us consider an entirely abstract case, namely, that N_1/N is a function of δ only. Assuming for simplicity that within a limited range of δ we have

$$\frac{N_1}{N} \propto \delta, \tag{1}$$

let us see what applications could be made of such a relation if it actually existed.

I

Let the active class be composed principally of two types of individuals: the military and the industrial. Let their corresponding numbers be x_0 and y_0. Thus

$$x_0 + y_0 = N_1. \tag{2}$$

From expression (1) we have

$$\frac{x_0 + y_0}{N} = a\,\delta, \tag{3}$$

where a is a factor of proportionality.

On the other hand, we may consider that the greater the ratio y_0/N, the greater the ability to produce goods. For two nations

with the same y_0/N the one which has more natural resources per capita will produce more goods per capita. Let us put again, as in chapter v, approximately the total amount of natural resources as proportional to the area S. The per capita amount of resources is then S/N. The per capita production per unit amount of S/N is therefore proportional to y_0/N. Hence, denoting by i the per capita production and by b and c two other coefficients, we have

$$\frac{i}{bS/N} = c\frac{y_0}{N}, \tag{4}$$

or, putting

$$\frac{1}{b\,c} = g, \tag{5}$$

$$y_0 = g\,i\,\delta\,N. \tag{6}$$

Equations (3) and (6) give:

$$\frac{x_0}{y_0} = \frac{A-i}{i}, \tag{7}$$

where

$$A = \frac{a}{g}. \tag{8}$$

The ratio x_0/y_0, which determines the composition of the total active class, affects a number of relations in the social group. The greater x_0/y_0, the more, for instance, will the military activities prevail over the peaceful industrial activities. We should in general expect that the larger x_0/y_0, the larger fraction of the total income of a nation would be spent for military purposes. If there is a nation for which $x_0/y_0 = 0$, then for that nation equation (7) gives:

$$i = i_0 = A. \tag{9}$$

Relation (9) will hold approximately even when x_0/y_0 is not zero, but very small. With this degree of approximation we may consider that in the pre-war United States x_0/y_0 was very small, and put A equal to the yearly per capita income in the United States before the war.

The greater x_0/y_0, the greater in general will be the fraction μ_w of the total national income which is spent for war purposes in time of peace.

While only further theoretical studies could establish a relation between x_0/y_0 and the fraction μ_w, yet a general property of that relation can be deduced at once. Since $\mu_w < 1$, therefore the curve rep-

resenting μ_w as a function of x_0/y_0 should increase monotonically, but tend asymptotically to 1. It must therefore be convex upward.

It may be seen on general principles that the relation between μ_w and x_0/y_0 must also contain i as a parameter. The greater i, the larger percentage μ_w may be taken away from a person's income, still leaving him enough for his needs. However, even for infinite i, μ_w will probably remain substantially less than one.

Within a range of small values of μ_w we may expect that μ_w will vary as some power p of i, with $p < 1$, and approximately linearly with x_0/y_0. Hence, approximately:

$$\mu_w \propto i^p \, x_0/y_0 , \tag{10}$$

or, combining this with equation (7):

$$\mu_w = B \, i^{p-1} \, (A - i), \tag{11}$$

B being a coefficient. Figure 1 shows the comparison of equation

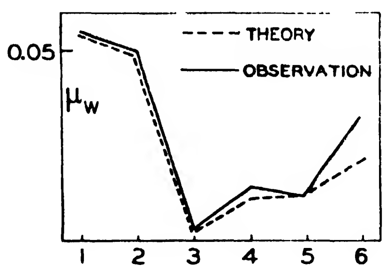

FIGURE 1
1—Japan; 2—Russia; 3—Canada; 4—Switzerland; 5—Norway; 6—Finland.

(11) with available data[1] for approximately 1929, with $B = 0{\cdot}063$, $A = 640$ dollars per person per year and $p = 1/2$.

Great Britain and France have not been included in this illustration for the same reason as in the discussion on page 63 of chapter vi, namely, because the presence of large colonies requires a change in the method of calculations.

Equation (6) may also be subjected to a test in the following manner. The number I of discoveries and inventions in a country for a given period are approximately proportional to y_0. Hence

$$I = C \, i \delta \, N , \tag{12}$$

C being a constant. A comparison of equation (12) with available

data is shown on Figure 2. The data for I are taken from Sorokin[2], and refer approximately to the end of the 19th Century.

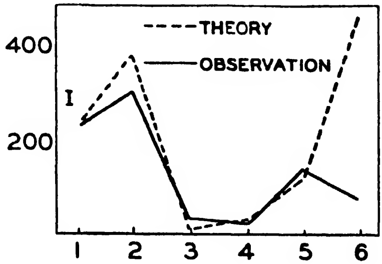

FIGURE 2
1—United States; 2—Germany; 3—Russia; 4—Switzerland; 5—France; 6 —England.

If we compute from equation (12) the values of I for France and England, we find a correct value for the former, but a much too low value for the latter. This probably is due, as in chapter v, page 63, to the neglect of the effect of the colonies. The too low value for England is to be expected; the correct value for France may be accidental.

As has been emphasized frequently on previous occasions, in such cases we should not put too much weight upon the agreement between theory and observations. The theory in the present form is too crude and available data both scarce and inaccurate. In general one would expect x_0/y_0 to be determined by many other factors besides i. Expression (7) owes its simplicity principally to the simplifying assumption (1), as well as to the simplifying assumption that the amount R of natural resources is proportional to S. If instead of expressions (1) and (3) we put generally

$$\frac{x_0 + y_0}{N} = f(\delta),$$

and instead of equation (4) put

$$\frac{iN}{bR} = c\frac{y_0}{N},$$

we obtain instead of equation (7) the expression

$$\frac{x_0}{y_0} = \frac{bcf(\delta)\, R - iN}{iN},$$

which for $f(\delta) = a\delta$, $R = S$, reduces to (7).

We may here make a slight improvement in the argument used in chapter vii, page 68. We assumed there the quantity a to be very small, so as to make equation (13) of chapter vii contain only two variables, x_0/y_0 and ε_m. We shall now eliminate y_0 from equation (13) of chapter vii by means of equation (3) of the present chapter. Since $x_0 + y_0 << N$, we may put

$$N = N'; \quad x_0 + y_0 = \gamma' N'^2, \tag{13}$$

where γ' is a constant.

We may consider here as a theoretical possibility that γ' is approximately the same for different social groups. Still more in line with the fundamental assumptions would be to consider that

$$\frac{x_0 + y_0}{N'} = \gamma\,\delta, \tag{14}$$

where γ is again a constant, while $\delta = N'/S$ is the density of population.

Solving equation (13) of chapter vii and (14) of the present chapter for x_0 and y_0, we find

$$x_0 = \frac{c_0^*(1 + \varepsilon_m)\,\gamma N'\delta - aN'}{a_0^* + c_0^*(1 + \varepsilon_m)}; \quad y_0 = \frac{(a_0^*\delta\gamma + a)\,N'}{a_0^* + c_0^*(1 + \varepsilon_m)}, \tag{15}$$

or

$$\frac{x_0}{y_0} = \frac{c_0^*\gamma\delta - a}{a_0^*\gamma\delta + a} + \frac{c_0^*\gamma\delta}{a_0^*\gamma\delta + a}\,\varepsilon_m, \tag{16}$$

which is again of the form of equation (16) of chapter vii. Hence the results obtained from that equation need not be based on the assumption that a is very small. The different assumption used here may be substituted for it.

We may also consider that

$$\frac{x_0 + y_0}{N'} = \text{Const.} = \gamma''. \tag{17}$$

This leads to an expression similar to (16). All we have to do is to substitute in (16) γ'' for $\gamma\delta$.

Another approximate expression, used in chapter vii, does change, however, with the new assumptions. We refer to expression (14) and its approximation (19).

Introducing into (14) of chapter vii the expression (15) of this chapter for y_0, we find

$$\frac{y_1}{x_1 + y} = \frac{[\lambda c_0^*(1 + \varepsilon_1) - c_0^*(1 + \varepsilon_m)](a_0^*\delta\gamma + a)}{a c_0^*(a_0^*\delta\gamma + a)\varepsilon_1 - 2aa_0^* - 2ac_0^*(1 + \varepsilon_m)}, \tag{18}$$

which is of the form

$$\frac{y_1}{x_1 + y_1} = \frac{A - B\,\varepsilon_m}{C - D\,\varepsilon_m}. \tag{19}$$

II

We shall now briefly outline a method of generalizing the above considerations to countries with large colonies. Putting

$$\frac{y_0}{N} = \zeta, \tag{20}$$

we have, from equations (4) and (5):

$$i = \frac{\zeta}{g}\frac{S}{N}. \tag{21}$$

For a constant ζ the per capita production is proportional to S. Let the country have a colony with a population N' and area S', at an average distance r' from the metropolitan country. Let N denote the population of the metropolitan country only. Consider approximately that the population N' does not share in the income of the metropolitan country. In other words, i remains the per capita income of the metropolitan country only. Then this income i would be

$$\frac{\zeta}{g}\frac{S + S'}{N}, \tag{22}$$

less an amount proportional to the cost of work necessary to transport the natural resources of the colony to the metropolitan country, and less the amount necessary to give a minimum life subsistance to the population of the colony. The first term is proportional to $r'S'/N$; the second, to N'. Hence, denoting by k and k' two coefficients,

$$i = \frac{\zeta}{g}\left(\frac{S + S'}{N} - k'r'\frac{S'}{N} - kN'\right) \tag{23}$$

or

$$i = \frac{\zeta}{g}\frac{S + S'(1 - k'r') - kNN'}{N}. \tag{24}$$

Equation (24) becomes of the same form as equation (21) if we introduce the concept of "effective area" $\bar{S}$ given by

$$\bar{S} = S + S'(1 - kr') - kNN'. \tag{25}$$

The "effective density" of the population is

$$\bar{\delta} = \frac{\bar{S}}{N}. \tag{26}$$

More generally, for any number of colonies, we have, denoting by $S^{(i)}$ the area of the ith colony, by $N^{(i)}$ its population, and by $r^{(i)}$ its distance from the metropolitan country:

$$\bar{\Delta} = \frac{S + \sum_i S^{(i)}(1 - k^{(i)}r^{(i)}) - kn \sum_i N^{(i)}}{N}. \tag{27}$$

If we assume the constants k and k' to be the same for all countries, we may determine them in the following way.

If we know the values of i, δ and N for countries without colonies, we may determine the constant C in equation (12). For a country with colonies we have

$$I = Ci\bar{\delta}N. \tag{28}$$

Knowing I, we can determine $\bar{\delta}$. From any two countries with colonies we may then determine k and k' from equation (27).

These considerations again are more of a methodological than of a "practical" interest.

III

The discussions of the first section suggest a possible comparison of different countries with respect to their importance and influence in international relations. We can roughly estimate the relative sizes of the active groups for different countries by using the relation (3).

While within the same country two or more active groups may dispute domination over the passive population, in a collection of countries the dominating active groups of every country will try to impose their preferred type of behavior on the other countries. Even while following opposite tendencies in their own country, the active groups will in general have a common interest in the control of another country. The military group may do it out of its natural desire for domination, the industrial group may have considerations of more economic character. Thus neither purely military nor purely economic factors, but their combination, would come into play in in-

ternational relations. The relative importance of either of the two factors will be determined by the composition of the active classes, that is, by the ratio $x_0/(x_0 + y_0)$, or, if besides x_0 and y_0 there is also a group z_0 (chapter xxi), then by $x_0/(x_0 + y_0 + z_0)$ and $y_0/(x_0 + y_0 + z_0)$.

We may thus consider the interrelations between different countries as a more general case of interaction of active classes. We would therefore expect in general a country to be the more influential in international affairs, the larger its $x_0 + y_0$, a quantity which again we shall denote by N_1:

$$N_1 = x_0 + y_0 . \tag{29}$$

One might argue that the total "influence" of a country in international relations should be proportional not to N_1, but to

$$N_1 N = a\,\delta\,N^2 = \frac{aN^3}{S}, \tag{30}$$

since a given active group N_1 has the more power to control other groups, the larger the total number N of people at its command.

We may also take the view that the greater the population density of a country, the smaller its per capita resources and the stronger its tendency to expand.

It is interesting to follow the variations of the index N^3/S with respect to time for different nations. The results are shown very roughly[3] in Figure 3, which is rather suggestive. From the end of the middle ages up to about the 19th Century, France had the largest strength and was the predominant country. The 19th Century witnessed a rapid increase of the strength of Great Britain. By the second half of the 19th Century Great Britain became definitely predominant over France. Prussia began to be noticeable late in the 18th Century, catching up with France very rapidly and approximately equalling the latter by 1873. From then on we have to consider Germany as an entity, dominated however by Prussia whose value of N^3/S exceeds that of the rest of Germany. Russia entered the international arena rather late. During the Seven Years' War at the end of the 18th Century it was stronger than Prussia.

We do not suggest that the outcome of a war is determined by the above considerations. Such an outcome is determined by many other factors, to be discussed in the last two chapters. It is possible that the ratio of the quantities $P = N^3/S$ for two countries determines the *probability* of the outcome. One might try to develop a relation between this probability and the ratio P_1/P_2 for two countries. In this fashion one could perhaps arrive at an expression which would

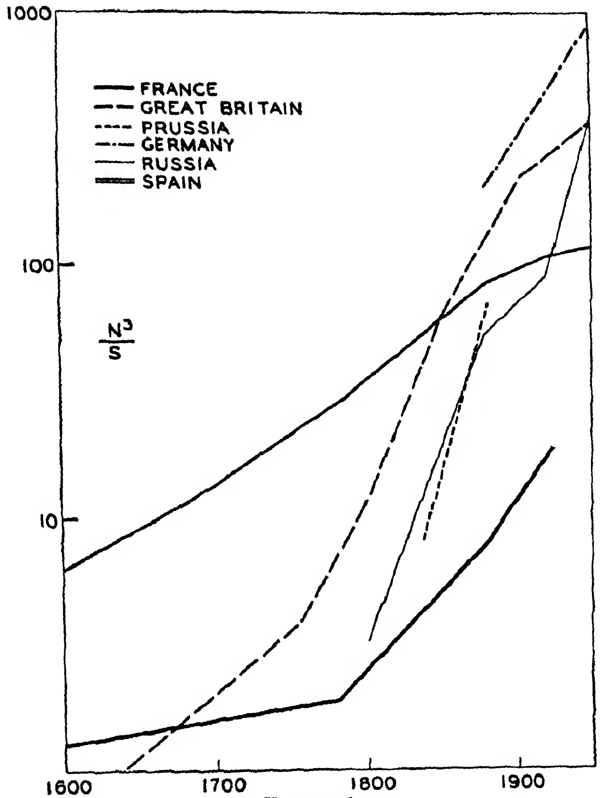

FIGURE 3

Ordinates on logarithmic scale. Explanation in text.

give us the percentage of wars lost by a given country during a given period to another country. This could be readily compared with available data.

It must be emphasized that the above does not constitute a theory, nor even a suggestion of a theory, of international history. It merely illustrates what could be obtained with appropriate quantitative indices.

Among other factors which determine the outcome of an international conflict is the rather elusive and vague "feeling of national unity." On the basis of the above consideration, a quantitative meaning may be given to this concept.

The "feeling of national unity" will be the stronger, the larger the relative number of active individuals who advocate that unity. If this relative number increases with time, full national unity will be reached after a threshold is exceeded. The greater the initial

ratio N_1/N , the smaller will be the population density for which N_1/N will exceed the threshold, and, *ceteris paribus,* the stronger will be the national unity of the nation. We may therefore take as a measure of national unity the inverse of the population density of a country at the period when it begins its existence as a nation. Data of that type are very meager but are available for some countries[3].

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CHAPTER XVIII

ON THE CHANGE IN DEGREE OF ORGANIZATION OF SOCIETY WITH TIME

In chapter xvi we discussed the variation with respect to time of the ratio N_1/N of the number of actives to the total number of individuals in a society. As has been seen previously, in order for the actives to control the behavior of society, this ratio must exceed a certain threshold value (chapter iii). We actually considered hitherto the competition for control between the opposing active groups. But those results apply also to the case where there is only one active group.

We have considered hitherto essentially two different cases: that in which the efforts are constant, and that in which they are functions of achieved success. In the first instance we find that an active group of x_0 individuals imposes its behavior on all the passive population if, in previous notations,

$$x_0 > \frac{a}{a_0 + a} N + \frac{c_0 - a}{c_0 + a} y_0. \tag{1}$$

In particular we may consider that $y_0 = 0$, in which case, putting $x_0 = N_1$, inequality (1) reduces to

$$N_1 > \frac{a}{a_0 + a} N. \tag{2}$$

In this case behavior B is anything which is *not* behavior A, and which the passive population may do in the absence of any influence of the group x_0. Behavior A may stand for organized behavior, B for disorganized behavior. Equation (2) indicates that in order for an active group to control completely the behavior of the population, N_1/N must exceed the threshold

$$\frac{a}{a_0 + a}. \tag{3}$$

If inequality (2) is not satisfied, the whole society may exhibit either behavior A or B. If (2) is satisfied, it exhibits A.

For the case of efforts depending on the achieved success, we

have a similar situation. In general in this case a part of the society, consisting of x individuals, exhibits behavior A, another part, consisting of y individuals, exhibits behavior B. For this to happen, however, inequalities (27) and (28) of chapter iii must be satisfied. For $y_0 = 0$, $x_0 = N_1$, these inequalities reduce correspondingly to

$$N_1 > \frac{a}{a_0^* + a} N, \tag{4}$$

and

$$N_1 > \frac{2a}{a_0^* \varepsilon}. \tag{5}$$

If neither inequality (4) nor (5) is satisfied, society may again exhibit either all behavior A or all behavior B. If inequality (4) is satisfied but (5) is not, we have all behavior A; if (4) is not satisfied but (5) is, we have all behavior B. With both (4) and (5) satisfied, we find that $x > N$. Since physically x cannot exceed N, this means that actually x will be equal to N. Hence, if $y_0 = 0$, the essential condition for full control is inequality (4), and N_1/N must exceed the threshold

$$\frac{a}{a_0^* + a}. \tag{6}$$

Consider case 1 of chapter xvi. From equations (22) and (8) of that chapter we easily find that the value N_1/N tends to α, when N tends to zero. On the other hand, when N has practically reached its asymptotic value $N_\infty = A$ (page 133), we may substitute that value for N in equation (6) of chapter xvi. We then find that N_1 tends asymptotically, for case 1, to $2\alpha N_\infty$. Hence, for case 1 of chapter xvi, we have

$$\left(\frac{N_1}{N}\right)_0 = \alpha; \left(\frac{N_1}{N}\right)_\infty = 2\alpha. \tag{7}$$

By a similar reasoning, we find from equations (30), (6), and (8) of chapter xvi, for case 2,

$$\left(\frac{N_1}{N}\right)_0 = \frac{4\alpha}{3}; \left(\frac{N_1}{N}\right)_\infty = 2\alpha. \tag{8}$$

Denoting by h the threshold which N_1/N must exceed in order to make the full control by the actives possible, we have a society organized from its very beginning if

$$h < \left(\frac{N_1}{N}\right)_0. \tag{9}$$

If

$$\left(\frac{N_1}{N}\right)_0 < h < \left(\frac{N_1}{N}\right)_\infty, \tag{10}$$

then the society will be disorganized until the time t^*, which is given by the root of the equation

$$\frac{N_1(t)}{N(t)} = h, \tag{11}$$

where for $N_1(t)$ and $N(t)$ we substitute the corresponding expressions from chapter xvi.

If

$$\left(\frac{N_1}{N}\right)_\infty < h, \tag{12}$$

the society will never be organized.

We may consider the case, where the general birth and death rate coefficients a, b and c depend on whether or not the actives control the behavior of the society. Thus, under the control of the actives, a may increase, while b and c decrease. In that case sometime after the moment t^*, a sudden increase in the rate of growth of N will occur, and the variation of N_1/N will now follow a different equation. If b and c become equal to b_1 and c_1, then N_1/N may remain constant. It is perhaps possible that N_1/N will even decrease, resulting eventually in a loss of control and an increased b and c. Thus periodical fluctuations of organization of a society may be possible.

Let there now be *two* active groups, composed of N_1 and N_2 individuals, correspondingly, and N_3 passive individuals, so that $N_1 + N_2 + N_3 = N$. Again let $N_1 << N$, $N_2 << N$, so that we shall have

$$\begin{aligned} \frac{dN_1}{dt} &= \alpha_1 aN - (b_1 + c_1 N)\, N_1; \\ \frac{dN_2}{dt} &= \alpha_2 aN - (b_2 + c_2 N)\, N_2. \end{aligned} \tag{13}$$

Suppose that $b_1 < b$; $c_1 < c$, but that $b_2 = b$, $c_2 = c$. By an argument similar to the one which leads to the second expression (7), we now find that N_1/N will tend to the value

$$\left(\frac{N_1}{N}\right)_\infty = \frac{a}{b_1 + c_1 N_\infty}\alpha_1 = \frac{a}{b_1 + (a-b)(c_1/c)}\alpha_1 = \alpha_{1\infty} > \alpha_1. \tag{14}$$

The last inequality (14) can be seen to hold in the following way: If it were not true, then $b_1 + (a-b)\, c_1/c \geqq a$. Since $b_1 \leqq b$, there-

fore *a fortiori* $b + (a - b)c_1/c \geqq a$. This gives $b\left(1 - \frac{c_1}{c}\right)$ $\geqq a\left(1 - \frac{c_1}{c}\right)$, which is impossible if $a > b$ and $c_1 > 0$.

On the other hand, if $N_2 = 0$ for $N = 0$, then N_2/N will always be equal to α_2, as is readily seen from inspection of the second equation (13) and of equation (2) of chapter xvi. If $N_2 = \alpha_2 N$, then the second equation (13) is automatically satisfied if equation (2) of chapter xvi holds.

Let the effect of group N_1 be such that when it controls the behavior of society, this reduces b and c, and therefore also b_2 and c_2, to b_1 and c_1. This will happen when that class, N_1, consists of scientists, etc. For the control of the society by class I we must have [c.f. inequality (1)]

$$N_1 > h_1 N + h_2 N_2,$$

or

$$\frac{N_1}{N} > h_1 + h_2 \frac{N_2}{N}, \tag{15}$$

while for the control by class II we must have

$$N_2 > h_1' N + h_2' N_1 \quad \text{or} \quad \frac{N_2}{N} > h_1' + h_2' \frac{N_1}{N}, \tag{16}$$

where h_1, h_2, h_1' and h_2' are constants.

Let

$$\left(\frac{N_1}{N}\right)_0 = \alpha_{10} = \alpha_1. \tag{17}$$

Further, let

$$\alpha_2 > h_1' + h_2' \alpha_{10}\,; \quad \text{and} \quad \alpha_2 < h_1' + h_2' \alpha_{1\infty} \tag{18}$$

but

$$\alpha_{1\infty} > h_1 + h_2 \alpha_2\,. \tag{19}$$

In that case, while N is small and N_1/N is close to α_{10}, class II will be the controlling one, because inequality (16) will hold. As N increases, however, N_1/N increases to $\alpha_{1\infty} > \alpha_{10}$, while N_2/N remains constant. Hence at some time t_1 inequalities (15) will begin to hold. At $t = t_1 + \tau_1$ the constants b and c will become equal to b_1 and c_1 respectively and N_1/N will begin to decrease, tending to $\alpha_1 < \alpha_{1\infty}$. Therefore, because of inequalities (18), at some moment $t = t_2$, inequality (16) will again begin to hold, and the control passes to class II again. At $t_2 + \tau_2$, a and b will regain their initial values, N_1/N again

increases toward $\alpha_{1\infty}$, and eventually equation (15) again begins to hold at $t = t_1'$. Thus we have fluctuations of control. The times t_1, t_2, t_1', etc. are readily obtained for any of the cases studied before by adjusting the integration constant each time, so that, for instance, at $t = t_1$, $N_1/N = h_1 + h_2\alpha_2$, etc. The fluctuations in general will not be periodical, the intervals $t_2 - t_1$, $t_1' - t_2$, $t_2' - t_1'$ not being equal. As an example we may identify behavior *I* with Sorokin's "sensate," *II* with his "ideational," behavior[1]. For efforts varying with success, transitory "idealistic" stages will be obtained.

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CHAPTER XIX

GENERAL MATHEMATICAL THEORY OF INDIVIDUALISTIC AND COLLECTIVISTIC SOCIETIES

In chapter vi we applied the concept of satisfaction to the interaction of classes. The same concept may be applied to the interaction of two or more individuals. In all such cases of interaction, determined by maximizing the satisfaction function, there are fundamentally two different possibilities. Either each individual behaves so as to maximize his own satisfaction, or else he tries to maximize the sum total of satisfactions of all individuals. We may call the first type of behavior "egoistic" or "individualistic"; the second, "altruistic" or "collectivistic." It may be asked whether the assumption of the two different behaviors leads to different results, and, if so, under what conditions. The problem is in some respects similar to the problem of cooperation and competition in mathematical economics.[1] It is more general, however.

It is generally admitted that our present society is based on individualistic behavior. J. Rueff[2] considers the "collectivistic" behavior as comparable to non-euclidean geometrics in mathematics. The two types of behavior may be equally real psychologically, however, just as non-euclidean geometry became real in modern physics. Our problem is to study both first *in abstracto*. Later on it may be possible to apply the mathematical conclusions to actual cases.

A "collectivistic" behavior, fundamentally, puts the "society as a whole" above the individual. The connection between "collectivistic" principle and various systems of socialism, government control, etc. is apparent. Mathematical studies of different aspects of "individualistic" and "collectivistic" behaviors may throw some light upon the contemporary trends in social changes.

We shall first investigate the interaction of two individuals, and then generalize the results to the case of N individuals.

I

Let an individual perform an amount of work y, producing an amount of goods $x = ay$. Let the satisfaction function S with re-

spect to x be of the form $A \log \alpha x$. We shall assume that S decreases linearly with increasing y. This amounts to assuming that work is always unpleasant. Actually, the variation of S with y is more likely to be such that it first increases to a positive maximum, then decreases, becoming negative. Such an assumption would complicate our calculations. The more restrictive assumption, made here, is not likely to introduce serious limitations. We thus have

$$S = A \log \alpha x - By , \tag{1}$$

where A and B are constants. Because of $x = ay$, we have

$$S = A \log \alpha a y - By . \tag{2}$$

This has a maximum for

$$y = A/B . \tag{3}$$

If the individual tries to increase his satisfaction function to a maximum, he will perform the amount $y = A/B$ of work, and produce the amount $x = aA/B$ of goods.

Consider two individuals, with satisfaction functions of the same form, performing, respectively, the amounts y_1 and y_2 of work, and producing the amounts

$$x_1 = a_1 y_1 ; \quad x_2 = a_2 y_2 \tag{4}$$

of goods when working separately. Their satisfaction functions are

$$S_1 = A_1 \log \alpha_1 x_1 - B_1 y_1 ; \tag{5}$$

$$S_2 = A_2 \log \alpha_2 x_2 - B_2 y_2 . \tag{6}$$

When the individuals work on the production of the same goods together, their productivity in general increases. The total amount X of goods produced will be greater than the sum of $a_1 y_1 + a_2 y_2$. Depending on the nature of the work and on the type of cooperation, we shall have different expressions for X. We shall here consider one of the simplest possible assumptions, namely,

$$X = a_1 y_1 + a_2 y_2 + a_3 y_1 y_2 . \tag{7}$$

For $y_2 = 0$, this reduces to $a_1 y_1$; for $y_1 = 0$ it reduces to $a_2 y_2$.

The total amount X of goods produced may be distributed in different ways between the two individuals. If they apply the principle that a person receives in proportion to the *amount* of work he does, then the total amount X will be divided between the two individuals in the ratio y_1/y_2. If a person receives in proportion to the *effectiveness* of his work, then the total amount X will be divided in

the ratio a_1y_1/a_2y_2. Finally, the division may be made in a fixed ratio, independent of the relative amounts of work. This ratio may in particular be equal to 1.

We shall first consider the second case, that of remuneration in proportion to effectiveness of work. The first case is handled in a similar way and leads to similar results.

Denoting now by x_1 and x_2 the corresponding share of each individual, so that $X = x_1 + x_2$, we have:

$$\frac{x_1}{x_1 + x_2} = \frac{a_1y_1}{a_1y_1 + a_2y_2}. \tag{8}$$

Because of equation (7) we have $x_1 + x_2 = a_1y_1 + a_2y_2 + a_3y_1y_2$; and this, together with equation (8), gives

$$x_1 = \frac{a_1y_1 + a_2y_2 + a_3y_1y_2}{a_1y_1 + a_2y_2}\,a_1y_1; \tag{9}$$

$$x_2 = \frac{a_1y_1 + a_2y_2 + a_3y_1y_2}{a_1y_1 + a_2y_2}\,a_2y_2. \tag{10}$$

Introducing those expressions into equations (5) and (6), we obtain:

$$S_1 = A_1 \log \alpha_1 \frac{a_1y_1 + a_2y_2 + a_3y_1y_2}{a_1y_1 + a_2y_2}\,a_1y_1 - B_1y_1; \tag{11}$$

$$S_2 = A_2 \log \alpha_2 \frac{a_1y_1 + a_2y_2 + a_3y_1y_2}{a_1y_1 + a_2y_2}\,a_2y_2 - B_2y_2. \tag{12}$$

The amount of work which each individual will perform will now be determined by his attitude towards the satisfaction function. Two possibilities exist.

1) Each individual tries to maximize his own satisfaction function, regardless of the other individual. In this case the values of y_1 and y_2 are determined from

$$\frac{\partial S_1}{\partial y_1} = 0; \quad \frac{\partial S_2}{\partial y_2} = 0. \tag{13}$$

2) Each individual tries to maximize the total satisfaction $S = S_1 + S_2$. In this case y_1 and y_2 are determined from

$$\frac{\partial S}{\partial y_1} = 0; \quad \frac{\partial S}{\partial y_2} = 0. \tag{14}$$

From equation (11) we have:

$$\frac{\partial S_1}{\partial y_1} = \tag{15}$$
$$A_1 \frac{a_1^3 y_1^2 + a_1^2 a_3 y_1^2 y_2 + 2a_1^2 a_2 y_1 y_2 + a_1 a_2^2 y_2^2 + 2a_1 a_2 a_3 y_1 y_2^2}{(a_1 y_1 + a_2 y_2)(a_1 y_1 + a_2 y_2 + a_3 y_1 y_2)\, a_1 y_1} - B_1,$$

and from equation (12) we find a similar expression for $\partial S_2/\partial y_2$.

Introducing expression (15) and a corresponding one for $\partial S_2/\partial y_2$ into equations (13) gives us two cubic equations in y_1 and y_2, whose solution is impractical. We can, however, investigate the properties of the solution without actually solving the equations.

Put

$$\frac{\partial S_1}{\partial y_1} = F_1(y_1, y_2), \tag{16}$$

and consider the curve determined by

$$F_1(y_1, y_2) = 0. \tag{17}$$

If $y_2 = 0$, then, from equation (15), we have

$$F_1(y_1, 0) = \frac{A_1}{y_1} - B_1. \tag{18}$$

Hence, for $y_2 = 0$, the requirement $F_1(y_1, y_2) = 0$ leads to

$$y_1 = \frac{A_1}{B_1}. \tag{19}$$

Thus the curve $F_1(y_1, y_2) = 0$ intersects the y_1-axis at the point given by equation (19) (Figure 1, full line).

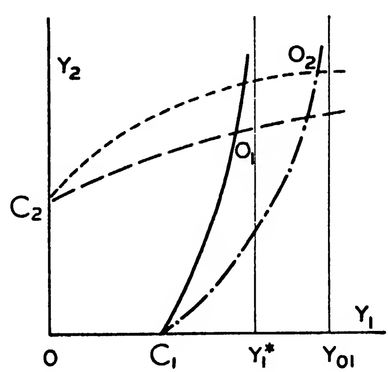

FIGURE 1

When $y_2 = \infty$, then, from equations (15) and (16), we have

$$F(y_1, \infty) = A_1 \frac{a_1 a_2^2 + 2a_1 a_2 a_3 y_1}{(a_1 a_2^2 + a_1 a_2 a_3 y_1) y_1} - B_1. \tag{20}$$

Putting $F_1(y_1, \infty) = 0$ and introducing the notation:

$$\frac{A_1}{B_1} = C_1, \tag{21}$$

gives

$$a_1 a_2 a_3 y_1^2 + (a_1 a_2^2 - 2C_1 a_1 a_2 a_3) y_1 - C_1 a_1 a_2^2 = 0. \tag{22}$$

Equation (22) has two roots:

$$\begin{aligned} y_1^* &= \frac{-a_1 a_2^2 + 2C_1 a_1 a_2 a_3 + \sqrt{a_1^2 a_2^4 + 4(C_1 a_1 a_2 a_3)^2}}{2a_1 a_2 a_3}; \\ y_1^{**} &= \frac{-a_1 a_2^2 + 2C_1 a_1 a_2 a_3 - \sqrt{a_1^2 a_2^4 + 4(C_1 a_1 a_2 a_3)^2}}{2a_1 a_2 a_3}. \end{aligned} \tag{23}$$

Since $y_1^* > 0$, $y_1^{**} < 0$, the second root has no physical meaning, and we therefore consider only y_1^*. Hence, the curve $F_1(y_1, y_2) = 0$ is such that $y_2 = \infty$ for $y_1 = y_1^*$. We have, from the first equation (23):

$$y_1^* - C_1 = \frac{-a_1 a_2^2 + \sqrt{(a_1 a_2^2)^2 + 4(C_1 a_1 a_2 a_3)^2}}{2a_1 a_2 a_3} > 0. \tag{24}$$

Hence (cf. Figure 1):

$$C_1 < y_1^*. \tag{25}$$

We shall now prove that the curve $F_1(y_1, y_2) = 0$ is monotonic. To this end we must establish that $dy_1/dy_2 > 0$. We have

$$\frac{dy_1}{dy_2} = -\frac{\dfrac{\partial F_1}{\partial y_2}}{\dfrac{\partial F_1}{\partial y_1}}. \tag{26}$$

The function F_1 is given by the right side of equation (15). The derivative $\partial F_1/\partial y_1$ is of the form D_1/D, where D is the square of the denominator in equation (15) and is therefore always positive. The derivative $\partial F_1/\partial y_2$ is of the form D_2/D. Evaluating D_1 and D_2, we find after laborious but elementary calculation that D_1 consists of a sum of only negative terms, while D_2 consists of a sum of positive terms. Hence, $D_1 < 0$, $D_2 > 0$. Therefore $\partial F_1/\partial y_2 < 0$; $\partial F_1/\partial y_1 > 0$.

Hence, from equation (26) it follows that $dy_1/dy_2 > 0$.

By similar reasoning we determine the general properties of the curve $\partial S_2/\partial y_2 = F_2(y_1, y_2) = 0$ (Figure 1, broken line). The two curves intersect at only one point, O_1. The coordinates y_1 and y_2 of O_1 represent the solution of the system (13).

Now let us investigate case 2, which leads to equations (14).

From equations (9) and (10) we have:

$$\frac{\partial x_1}{\partial y_2} = \frac{a_1^2 a_3 y_1^2}{(a_1 y_1 + a_2 y_2)^2} > 0; \quad \frac{\partial x_2}{\partial y_1} = \frac{a_2^2 a_3 y_2^2}{(a_1 y_1 + a_2 y_2)^2} > 0. \tag{27}$$

Hence

$$\frac{\partial S_1}{\partial y_2} = \frac{A_1}{a_1 x_1}\frac{\partial x_1}{\partial y_2} > 0; \quad \frac{\partial S_2}{\partial y_1} = \frac{A_2}{a_2 x_2}\frac{\partial x_2}{\partial y_1} > 0. \tag{28}$$

Equations (14) may be written:

$$\frac{\partial S_1}{\partial y_1} + \frac{\partial S_2}{\partial y_1} = 0; \quad \frac{\partial S_1}{\partial y_2} + \frac{\partial S_2}{\partial y_2} = 0. \tag{29}$$

Since [equation (28)] $\partial S_2/\partial y_1 > 0$, therefore $\partial S/\partial y_1 > 0$ at all points of the full curve of Figure 1. Since for sufficiently large values of y_1, $\partial S_2/\partial y_1$ tends to zero, as seen from equations (27) and (28), while $\partial S_1/\partial y_1$ tends to a negative constant, therefore the locus of the maxima of S will be to the right of the full line (alternating line). By a similar argument we prove that the locus of $\partial S/\partial y_2 = 0$ is given by the dotted line of Figure 1. The values of y_1 and y_2, which correspond to a maximum of $S = S_1 + S_2$, are now given by the point O_2. Hence, in this case y_1 and y_2, and therefore x_1 and x_2, are greater than in the previous case.

That the line $\partial S/\partial y_1 = 0$ intersects the y_1-axis at the point C_1 is shown by the following considerations.

Put

$$\frac{\partial S}{\partial y_1} = U_1(y_1, y_2) = F_1(y_1, y_2) + V_1(y_1, y_2); \tag{30}$$

$$\frac{\partial S}{\partial y_2} = U_2(y_1, y_2) = F_2(y_1, y_2) + V_2(y_1, y_2); \tag{31}$$

where

$$V_1(y_1, y_2) = \frac{\partial S_2}{\partial y_1}; \quad V_2(y_1, y_2) = \frac{\partial S_1}{\partial y_2}. \tag{32}$$

From equation (12) we have:

$$V_1(y_1, y_2) = A_2 \frac{a_2 a_3 y_2^2}{(a_1 y_1 + a_2 y_2)(a_1 y_1 + a_2 y_2 + a_3 y_1 y_2)}. \tag{33}$$

For $y_2 = 0$, $V_1 = 0$; showing that the curves $U_1 = 0$ and $F_1 = 0$ intersect the y_1-axis at the same point.

Similarly, we prove that the broken and dotted lines of Figure 1 intersect the y_2-axis at the same point, $C_2 = A_2/B_2$.

Now we proceed to the discussion of the case where the total amount $x_1 + x_2$ of goods is divided in a fixed ratio, so that

$$\begin{aligned} x_1 &= \beta(x_1 + x_2); \\ x_2 &= (1-\beta)(x_1 + x_2). \end{aligned} \tag{34}$$

From equations (34) and (7) it follows that:

$$\begin{aligned} x_1 &= \beta(a_1 y_1 + a_2 y_2 + a_3 y_1 y_2); \\ x_2 &= (1-\beta)(a_1 y_1 + a_2 y_2 + a_3 y_1 y_2). \end{aligned} \tag{35}$$

From equations (5), (6) and (35), we now obtain:

$$\frac{\partial S_1}{\partial y_1} = F_1(y_1, y_2) = A_1 \frac{a_1 + a_3 y_2}{a_1 y_1 + a_2 y_2 + a_3 y_1 y_2} - B_1; \tag{36}$$

$$\frac{\partial S_2}{\partial y_1} = V_1(y_1, y_2) = A_2 \frac{a_1 + a_3 y_2}{a_1 y_1 + a_2 y_2 + a_3 y_1 y_2} > 0. \tag{37}$$

The requirement $F_1(y_1, y_2) = 0$ gives

$$y_2 = \frac{a_1(y_1 - C_1)}{C_1 a_3 - a_2 - a_3 y_1}. \tag{38}$$

When $y_1 = C_1$, then $y_2 = 0$; while when

$$y_1 = \bar{y}_1^* = \frac{C_1 a_3 - a_2}{a_3} = C_1 - \frac{a_2}{a_3}, \tag{39}$$

then $y_2 = \infty$.

From equation (39) it follows that

$$\bar{y}_1^* < C_1. \tag{40}$$

Hence, the curve $F_1(y_1, y_2) = 0$ is such as shown in Figure 2 by the full line.

Expression (37) shows that $\partial S_2/\partial y_1 > 0$. Similarly, we find that $\partial S_1/\partial y_2 > 0$. Therefore, as in the preceding case, we find that the curve $\partial S/\partial y_1 = 0$ will be to the right of the curve $\partial S_1/\partial y_1 = 0$ (Figure 2, alternating line), while the curve $\partial S/\partial y_2 = 0$ will be above the curve $\partial S_2/\partial y_2 = 0$. Hence, maximizing S, rather than S_1 and S_2 separately,

results again in an increase of y_1 and y_2, and therefore of x_1 and x_2.

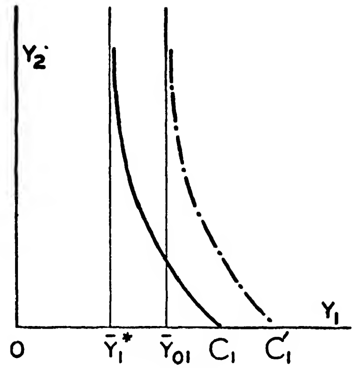

FIGURE 2

Consider

$$\frac{\partial S}{\partial y_1} = U_1(y_1, y_2) = F_1(y_1, y_2) + V_1(y_1, y_2). \tag{41}$$

It must be remembered that the functions $U_1(y_1, y_2)$ and $V_1(y_1, y_2)$ are different in the present case, which corresponds to a division of goods in a fixed ratio, from what they are in the case which corresponds to a division in proportion to the efforts. We have from equations (36) and (37):

$$U_1 = (A_1 + A_2)\frac{a_1 + a_3 y_2}{a_1 y_1 + a_2 y_2 + a_3 y_1 y_2} - B_1. \tag{42}$$

Put

$$\frac{A_1 + A_2}{B_1} = C_1' = C_1 + \frac{A_2}{B_1} > C_1. \tag{43}$$

Equation $U_1(y_1, y_2) = 0$ now gives

$$y_2 = \frac{a_1(y_1 - C_1')}{C_1' a_3 - a_2 - a_3 y_1}. \tag{44}$$

When $y_2 = 0$, then, because of (43), $y_1 = C_1' > C$; when

$$y_1 = \bar{y}_{01} = C_1' - \frac{a_2}{a_3} < C_1', \tag{45}$$

then $y_2 = \infty$.

Denote by y_{01} the value of y_1 for which y_2 of the curve $U_1(y_1, y_2) = 0$ in Figure 1 (alternating line) becomes infinite. The value of y_{01} is given by the equation obtained by taking the sum of the right sides

of equations (15) and (33), setting there $y_2 = \infty$, and *equating* the result to zero. This gives a cubic equation in y_1:

$$A_1 \frac{a_2^2 + 2a_2a_3y_1}{(a_2^2 + a_2a_3y_1)y_1} - B_1 + A_2 \frac{a_2a_3}{a_2^2 + a_2a_3y_1} = 0. \qquad (46)$$

Equation (46) has at least one positive root because for $y_1 = 0$ the left side is equal to $+\infty$, while for $y_1 = \infty$, the left side is negative. But

$$\frac{a_2a_3}{a_2^2 + a_2a_3y_1} < \frac{a_2^2 + 2a_2a_3y_1}{(a_2^2 + a_2a_3y_1)y_1}. \qquad (47)$$

Hence, for the value y_{01} of y_1, which satisfies equation (46), we have:

$$(A_1 + A_2)\frac{a_2a_3}{a_2^2 + a_2a_3y_{01}} - B_1 < 0. \qquad (48)$$

Since inequality (48) holds also when y_{01} tends to infinity, therefore the value y_{01}' which satisfies

$$(A_1 + A_2)\frac{a_2a_3}{a_2^2 + a_2a_3y_1} - B_1 = 0 \qquad (49)$$

is less than y_{01}.

Because of equation (43), equation (49) becomes:

$$C_1'a_2a_3 - a_2^2 - a_2a_3y_1 = 0. \qquad (50)$$

Hence, because of equation (43),

$$y_{01}' = C_1' - \frac{a_2}{a_3} = C_1 + \frac{A_2}{B_1} - \frac{a_2}{a_3}. \qquad (51)$$

Therefore

$$y_{01} > C_1 + \frac{A_2}{B_1} - \frac{a_2}{a_3}. \qquad (52)$$

The first equation (23) gives

$$y_1^* = C_1 - \frac{a_2}{2a_3} + \sqrt{\frac{a_2}{4a_3^2} + C_1} > C_1. \qquad (53)$$

Equations (45) and (43) give

$$\bar{y}_{01} = C_1 + \frac{A_2}{B_1} - \frac{a_2}{a_3}. \qquad (54)$$

From equations (52), (54), and (39) we find

$$y_{01} > \bar{y}_{01} > \bar{y}_1^*. \qquad (55)$$

Nothing can be said as to whether y_1^* lies between y_{01} and $\bar{y}_{01}$ or between $\bar{y}_{01}$ and $\bar{y}_1^*$. All that can be said is that $y_{01} > y_1^* > \bar{y}_1^*$.

The net result is that whether we divide the products in proportion to the effectiveness of the work or in a fixed ratio, maximizing of the total S results in a greater total productivity than maximizing of the individual values of S_1 and S_2. Inequalities (55) show that the best result is obtained for the case of a division of the products in proportion to the effectiveness of the work.

It is important to point out that the point O_2 in Figure 1 corresponds not only to a greater value of S, but that the values of S_1 and S_2 at that point are also greater than at O_1, because both S_1 and S_2 are positive. This may at first seem paradoxical, the point O being determined by maximizing S_1 and S_2 separately. Figure 3 explains this

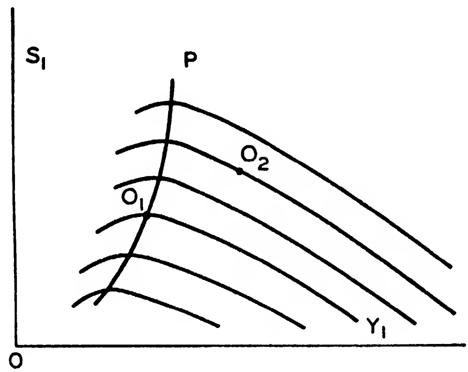

FIGURE 3

paradox. In this figure the lines of intersection of the S_1 surface with different planes $y_2 = \text{Const.}$ are shown. All maxima lie along the line P, given by $F_1(y_1, y_2) = 0$. In trying to maximize *his own* S_1, regardless of the other individual, the first one will always remain on the line P, on which O_1 is also located. Although the point O_2 is off the locus of the maxima of S_1 with respect to y_1, yet $S_1(O_2) > S_1(O_1)$. Thus by trying to maximize his own S_1 and S_2, each individual actually has a smaller value of S_1 or S_2 than if they both try to maximize S.

II

All the above arguments may be generalized to the case of any number N of individuals. Denote by the index i the quantities referring to the ith individual. Instead of equations (9) and (10) we now we shall have

$$x_i = \frac{\sum_k a_k y_k + \sum_k \sum_l b_{kl} y_k y_l}{\sum_k a_k y_k} a_i y_i = \left(1 + \frac{\sum_k \sum_l b_{kl} y_k y_l}{\sum_k a_k y_k}\right) a_i y_i . \tag{56}$$

For the ith individual we have

$$S_i = A_i \log \alpha_i x_i - B_i y_i . \tag{57}$$

When the total number N of individuals is very large, then the derivative of $\sum_k a_k y_k$ and of $\sum_k \sum_l b_{kl} y_k y_l$ with respect to any one y_i is very small, if those sums themselves remain finite. When N becomes very large, then we may replace the sums with integrals. Instead of a discontinuous index i we introduce a continuous parameter λ. Denoting by $N(\lambda)\,d\lambda$ the number of individuals whose λ is between λ and $\lambda + d\lambda$; introducing instead of the a_i's, a function $a(\lambda)$; and instead of b_{kl}'s, a function $b(\lambda, \lambda')$, we have

$$\sum_k a_k y_k = \int_0^\infty N(\lambda) a(\lambda) y(\lambda)\, d\lambda ; \tag{58}$$

$$\sum_k \sum_l b_{kl} y_k y_l = \int_0^\infty \int_0^\infty N(\lambda) N(\lambda') b(\lambda, \lambda') y(\lambda) y(\lambda')\, d\lambda d\lambda' . \tag{59}$$

The functional derivative of the integral in expression (58) with respect to a particular $y(\lambda_i)$ is equal to

$$N(\lambda_i) a(\lambda_i)\, d\lambda , \tag{60}$$

and is infinitesimal quantity. A similar consideration holds about the double integral in expression (59). Physically this is quite intelligible. If x_i depends on the y's of very many individuals, the change in the value of y of only one individual will not appreciably affect the value of x_i. Hence from expression (56), with great approximation,

$$\frac{\partial x_i}{\partial y_i} = a_i \left(1 + \frac{\sum \sum b_{kl} y_k y_l}{\sum a_k y_k}\right) . \tag{61}$$

Therefore equation (57) gives

$$\frac{\partial S_i}{\partial y_i} = \frac{A_i}{y_i} - B_i . \tag{62}$$

If each individual maximizes his own S_i, then the values of y_i's are found from N equations

$$\frac{\partial S_i}{\partial y_i} = 0 . \tag{63}$$

They are the coordinates of the point O in an N dimensional hyperspace. The point O is the point of intersection of the N hypersurfaces given by equations (63). If we have the condition that

$$\frac{\partial S_i}{\partial y_k} > 0, \quad (i \gtrless k) \tag{64}$$

and if $\partial S_i/\partial y_k$ tends to zero for large values of y_k, then by a similar argument as before, we can prove that the point O_1, which corresponds to the solution of the system

$$\frac{\partial S}{\partial y_i} = \frac{\partial S_i}{\partial y_i} + \sum_{k \neq i} \frac{\partial S_k}{\partial y_i} = 0, \tag{65}$$

lies further from the origin than the point O, and thus corresponds to larger values of y_k's. From what was said above, the values of $\partial S_k/\partial y_i$ will tend to zero for $N \to \infty$. But the expression $\sum_{k \neq l} (\partial S_k/\partial y_i)$ will remain finite.

For the case of two individuals, condition (64) follows from inequalities (27). It can, however, be seen from equation (56), that already for $N = 3$, $\partial x_i/\partial y_k$ is not always positive. Hence, in this case the conclusions of section I may not hold for N individuals.

Consider, however, for the case of two individuals the expressions

$$\begin{aligned} x_1 &= a_1 y_1 (1 + a_3 y_1 y_2); \\ x_2 &= a_2 y_2 (1 + a_3 y_1 y_2), \end{aligned} \tag{66}$$

The corresponding expression for N individuals is

$$x_i = a_i y_i (1 + \sum_k \sum_l b_{kl} y_k y_l). \tag{67}$$

Whereas expressions (9) and (10) correspond to the assumption (7), expression (66) corresponds to

$$x_1 + x_2 = a_1 y_1 + a_2 y_2 + a_1 a_3 y_1^2 y_2 + a_2 a_3 y_1 y_2^2. \tag{68}$$

The *total* amount of goods produced increases in expression (68) much more rapidly with the y_k's than in expression (7). In other words, expression (68), (66) and (67) correspond to the case where cooperative effort is more beneficial than in expression (7).

Expression (67) gives

$$\frac{\partial x_i}{\partial y_k} > 0, \quad (i \gtrless k). \tag{69}$$

Introducing this into equations (57), we find that now conditions (64) are satisfied and, moreover, that $\partial S_i/\partial y_k \to 0$ when $y_k \to \infty$.

Hence, if the nature of the production of goods is such that the total amount of goods produced *increases sufficiently strongly* with cooperation, maximizing of S leads to greater values of x's, y's, and of the S_i's, than maximizing the individual S_i's. A "collectivistic" principle will in this case lead to a greater satisfaction for each individual than will the "individualistic" principle.

The case of a division of goods in a constant ratio can be studied in a similar way. The comparison of the two cases for N individuals presents greater difficulties, however, and shall not be considered here.

It must be repeated again that the conclusions reached here hold only in so far as the initial assumptions hold. By changing the form of the satisfaction function we may obtain different results. We may ask for what forms of the satisfaction function these results will hold in general. Inasmuch as the form of the satisfaction function chosen above is psychologically plausible, the conclusions reached in this chapter possess the same amount of plausibility.

The more complex our industries become, the more they are benefited by proper cooperation. Some of them, in fact, could not be operated by a group of *absolutely independent* individuals. Thus, in manufacturing shoes it may be advantageous for 1,000 shoemakers to work in an organized way, with a proper division of labor. But it is also quite possible for each of them to work quite independently, so that each one performs all the operations necessary in the production of the shoes. Consider on the other hand the production of automobiles. While 1,000 individuals working in cooperation may produce them, it will be practically impossible for the same 1,000 individuals to produce automobiles if they work absolutely independently. In the case of automobiles the effect of cooperation is much more pronounced than in the case of shoes.

With the growth of such industries, in which cooperation is paramount, we should thereafter expect that a greater satisfaction for each individual will be obtained through the application of the "collectivistic" principle, in the sense defined at the beginning of this chapter, than through the application of the "individualistic" principle.

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CHAPTER XX

INDIVIDUALISTIC AND COLLECTIVISTIC SOCIETIES, CONTINUED

In the previous chapter we learned that under certain conditions which determine the effectiveness of cooperation, a "collectivistic" behavior leads to greater satisfaction to every individual than the "individualistic" behavior. In this chapter we shall discuss the problem from the point of view of the interaction of the active and passive classes. Such a study throws some light on the mechanism of possible transitions from one type of society to another.

It must be emphasized again that all the conclusions of this chapter hold only as long as the fundamental assumptions made here hold. Although these assumptions may appear rather plausible, nevertheless a number of factors which have been left out of consideration in this simplified, theoretical approach may alter the conclusions appreciably.

Consider an area S populated by a total of N individuals, of whom N_2 belong to class II, the organizing class, and N_3 belong to class III. We shall neglect class I at present and put $N_1 = 0$, so that

$$N_2 + N_3 = N. \tag{1}$$

As in chapter v, we shall again speak of classes II and III, and of individuals of type II and type III. An individual of type II is one who has organizing abilities. An individual of type III is passive. Class II is the class *originally* composed of individuals of type II; class III is the class originally composed of individuals of type III. With time, individuals of type III appear in class II and vice versa.

Let the individuals of class II organize a group of N'_3 individuals of class III to perform some productive work which otherwise the individuals of class III cannot perform as efficiently. If $N'_3 < N_3$, then

$$N''_3 = N_3 - N'_3 \tag{2}$$

individuals of class III remain unorganized. Each of them will produce an amount p_3 of goods for himself, and consume an amount c_3. In order that they may subsist, it is necessary that

$$\varepsilon_3 = p_3 - c_3 \geqq 0. \tag{3}$$

The relation between the N_2 individuals of class II and the N'_3 individuals of class III may be described by equations discussed in chap-

ter v. We shall add here, however, another important consideration which brings the situation closer to reality.

As the N individuals settle within the area S, we may, for simplicity, consider that each one of them receives, on the average, an amount

$$s=\frac{S}{N} \tag{4}$$

of land. This land is used not only for agricultural purposes, but for any kind of industries, such as mining, etc. It must be remembered that in the ultimate analysis any natural resource, used in any industry, comes from the land. In making the assumption (4), we disregard the possibility that some individuals may obtain areas of land with rich soil suitable for agricultural purposes; some others may obtain land with rich mineral deposits suitable for mining; still others may possess large water areas suitable for fishing, or areas suitable for hunting. Such disparities in the values of land possessed are of great importance, and simplification of the problem is the only excuse for neglecting them at present. In the future elaboration of the theory, such disparities will have to be taken into account. Even neglecting them, however, we shall obtain some rather interesting results.

However, a single individual can actually make use of only a limited amount of land. Let that amount be s_0. The quantity s_0 is determined by the individual's capacity for work and by the technical facilities at his disposal. With the improvement of the latter, s_0 will increase. But for *very small average densities of population*, such as were present in the early stages of human history,

$$s_0 << s=\frac{S}{N}. \tag{5}$$

Under those conditions an amount of land

$$S'_u=S-Ns_0 \tag{6}$$

remains free or unused.

We shall now consider a somewhat more general situation than the one discussed in chapters v and vii. In return for their "organizing services," let the individuals of class II not only retain a substantial fraction of goods produced by the N'_3 individuals working under their supervision, but also require *rights of possession* to the lands belonging to the N'_3 individuals. Actual history justifies the consideration of such an assumption. Moreover, it is psychologically a very natural one for if the individuals of class II have to control efficiently the utilization of the land possessed by the N'_3 individuals of

class *III*, they will naturally wish to have that control extended as far as actual property rights. On the other hand, if an individual of class *III* finds working under the organizing direction of individuals of class *II* is beneficial, so that after deduction of the fraction of goods which goes to class *II* he still is left with an amount much larger than p_3, then any such individuals will readily agree to transfer the property rights to his land to class *II*. At this stage, the acquisition of property may be regarded as a kind of compensation for the greater organizing abilities.

In general, it will also happen that when working under the supervision of an organizing class, each individual of type *III* will be able to take care of more land than when working unsupervised. Therefore, the amount of land for every individual of type *II* will be not $s_0(1 + N'_3/N_2)$, but

$$s_2 = s_{02} + s_0 \frac{N'_3}{N_2}, \tag{7}$$

where $s_{02} > s_0$.

Hence, while each of the N''_3 individuals remains in the possession of an area s, each of the N_2 individuals now possesses an amount given by equation (7).

Instead of equation (6) for the unused land, we now have

$$S_u = S - N_2 s_{02} - N_3 s_0 . \tag{8}$$

The quantity s_2 will be the greater, the greater the number of individuals of class *III* which the individuals of class *II* can organize and supervise. The ratio N'_3/N_2, and therefore the value of s_2, will be a function of the means of communications. With improvement of the latter, N'_3/N_2, and hence s_2, will increase. Thus N'_3/N_2 and s_2 are in general functions of the time t.

The N'_3 individuals of class *III* are now deprived of land property, but because of their work under the supervision of class *II*, they are better off than the N''_3 individuals of which each has an area s_0 of land.

With time, the total number N of individuals will increase. Let us consider the case of a constant profile society, so that with increasing N, the ratio N_2/N_3 remains constant. Because of the dissimilarity of progeny and parents, some of the progeny of class *II* will be of type *III*, while some of the progeny of class *III* will be of type *II*. If, as is usually the case in human society, the ownership of any land is hereditary, then eventually large portions of land will belong to individuals who are not capable of organizing its exploitation. On the other hand, amongst the progeny of the N'_3 individuals, whom we

shall refer to as class *III'*, there will appear individuals of type *II* capable of organizing work for other individuals of type *III*. If the individuals belonging to class *II* but being of type *III* do not relinquish their property rights to such individuals of type *II* but of class *III*, then a logical outcome for the latter is to use some of the free land S_u given by equation (8), and organize the work of some of the N''_3 individuals of class *III*, taking in return for that organization possession of their land. Thus S_u will decrease.

If we consider the case in which the population increases exponentially, then for a constant profile, denoting by N_{02} and N_{03} the initial values of N_2 and N_3, and by β a constant, we have from equation (8)

$$S_u(t) = S - N_{02}e^{\beta t}\, s_{02}(t) - N_{03}e^{\beta t}\, s_0(t). \tag{9}$$

As we said above, $s_{02}(t)$ and $s_0(t)$ increase in general with t. For the special case where $s_{02}(t) = s_{02} = \text{Const.}$, and $s_0(t) = s_0 = \text{Const.}$, the time t_0 at which S_u becomes zero is given by

$$t_0 = \frac{1}{\beta} \log \frac{S}{N_{02}s_{02} + N_{03}s_0}. \tag{10}$$

If, as was said above, the ratio N'_3/N_2 increases with time, then because of

$$\frac{N_3}{N_2} = \frac{N'_3}{N_2} + \frac{N''_3}{N_2} = \text{Const.}, \tag{11}$$

the ratio N''_3/N_2 decreases. Depending on the expression of N'_3/N_2 as a function of time, the ratio N''_3/N_2 may even become zero, either before or after S_u becomes zero.

For all times $t < t_0$ when $S_u > 0$, we have a condition of "equal opportunity" for any individual of type *II* regardless of whether he is born of parents of class *II* or *III*. During this period of time each individual of type *II* may interact with a given number of individuals of type *III* in a manner described in chapters v and vi. He will let every individual of type *III* work for him for such a fraction θ of goods produced which makes the profit of the "supervisor" or "organizer" a maximum. The usual equations of mathematical economics[1] may be used to describe the economic interaction of the different individuals of type *II*. It must be emphasized that the existence of a positive S_u provides a kind of large "class mobility." Any individual of type *II* born from parents in class *III* can himself become, due to $S_u > 0$, an individual of class *II*.

We also notice that under those conditions both the N_2 individuals of type *II* and the N'_3 individuals of type *III* stand to gain from

such an interaction. The only ones who do not gain anything are the N''_3 individuals of type *III*. In a social group with a sufficiently large ratio N_2/N_3, the number N''_3 will be zero. In that case, for $S_u > 0$ or $t < t_0$, the cooperation between individuals of type *II* and those of type *III* will result in an advantage for everybody, although the fundamental principle under which such a cooperation works will be the *maximizing of the rate of increase of wealth of class II*. That class, in the usual parlance, may be considered as the *capitalistic* class, and we thus see that for sufficiently large values of N_2/N_3 and for $t < t_0$ the "capitalistic" system results in a benefit for the social group as a whole, so that everyone is satisfied.

When the ratio N_2/N_3 is not sufficiently large, so that $N''_3 > 0$, then two things may happen:

a) If the net excess ϵ_3 of production over consumption for the N''_3 individuals is positive, those individuals will be able to continue existence as a "poor class." It is natural to assume that the net rate of reproduction of the N''_3 individuals, due to the poorer conditions of health, increased mortality, etc., will be less than that of the N_2 individuals of class *II* or of the N'_3 individuals of class *III*. In this case, the ratio N_3''/N'_3 will tend to zero. The "poor" class will become relatively less and less numerous. We may make different plausible assumptions about the dependence of the net rate of increase of population on the economic conditions, and then develop corresponding expressions for the decrease of the ratio N''_3/N'_3 with time.

b) If $\epsilon_3 < 0$, then the group of N''_3 "poor" individuals actually dies out.

Hence, if the initial conditions as given by N_{02} and N_{03} are such that t_0 is sufficiently large, then for that period of time a stable society will exist in which active class *II* interacts with a passive class *III* in such a way as to maximize the wealth or profit of class *II*, and this interaction will result in a benefit for *both* classes.

Things become different, however, for $t > t_0$ when $S_u = 0$. Now if an individual of type *II* is born in class *III*, he does not have the same opportunity as an individual of type *II* and class *II*. Due to the graded nature of the supervisory and organizational work, he may do some amount of the latter, but he will have no property rights to the land which is possessed by an individual of class *II*. Gradually a group *II'* of active individuals of type *II* belonging to class *III* will be formed. Whereas an essential part of the behavior of class *II* consists of maintaining the idea of the sacredness of property rights, the group *II'* will oppose this idea. Since all individuals of the group *II'* belong to class *III*, they will insist on the ownership rights for the whole class *III*. For the passive individuals of type *III* belonging to

class III, the existing interaction of classes is still beneficial, but for the group II' it is not. To break the existing status, the logical, and perhaps even more, the psychological, way is to propound some ideas of public ownership of natural resources as well as of means of production.

With time the number N_3'' of individuals of group II' increases, according to equations developed in chapter xiv. If we consider the case of constant coefficients of influence, then the time at which the group II' will gain control and impose its ideas on the whole society is given by equation (20), chapter xiv, with a corresponding change in notation. If, on the other hand, we have the situation described on page 32 of chapter iii, then to a certain extent the shift towards the new, "collectivistic" ideas is continuous. Denoting by x the number of passive individuals of class III who exhibit the behavior advocated by class II (sacredness of ownership), and by y the number of individuals who exhibit the behavior advocated by the group II', the ratio x/y is given by equation (34) of chapter iii in which we now put $x_0 = N_2$; $y_0 = N_3''$. We thus find:

$$\frac{x}{y} = \frac{a_0^* \dfrac{N_2}{N_3''} - \left[c_0^* (1 - \varepsilon' N_3') + \dfrac{a}{N_3''} N_3' \right]}{c_0^* - \dfrac{a}{N_3''} N_3' - a_0^* (1 - N'_3 \varepsilon) \dfrac{N_2}{N_3''}}. \tag{12}$$

Since from equations (5) to (15) of chapter xiv, N_2, N_3'', and N_3' are given as functions of time, equation (12) represents the spread of "collectivistic" ideologies, measured by the number of its followers, as a function of time.

Under the condition discussed above, the capitalistic system will become unstable after a certain time $t^* > t_0$, given by equation (20) of chapter xiv.

Thus the capitalistic system of private ownership and maximizing the rate of accumulation of wealth of the organizing class will be stable up to $t = t_0$. Since with an increase in used land the total amount of production of goods will increase as S_u decreases, the capitalistic system will reach its optimum conditions at $t = t_0$ when $S_u = 0$. From then until $t = t^*$, either a gradual social change of behavior and economic ideologies will take place, or at $t = t^*$ a sudden change will bring a substitution of a new type of economic ideology for the old one.

Let us, however, consider the case where any individual of type II' may, by a correspondingly greater effort in his productive work, accumulate sufficient wealth to buy property rights from individuals

of class *II*. This amounts to an unlimited mobility between the two classes. If the mobility goes in both directions, then individuals of type *III* born in class *II* will pass into class *III*, a phenomenon actually known in history. Let us consider the case where the principle of private enterprise and maximizing of individual profits holds. Then the N_2 individuals of type *II* and class *II* may form N_2 competing enterprises. We may apply to this case the equations used in mathematical economics. G. Evans[1] gives two somewhat different equations for the limiting number n of competitors, above which no profit is possible. We shall use Evans' notations except for putting $-a$ in place of his a, so that in our equations $a > 0$. Consider his equation (22) (Reference 1, page 32). We have

$$n \leqq \frac{A}{K}(b - Ba)^2. \tag{13}$$

The coefficients a and b determine the total demand y for a given price p; thus

$$y = -ap + b\,. \tag{14}$$

But for a given price the total demand is proportional to the total number of buying population, that is, roughly proportional to N. Hence, denoting by a_0 and b_0 two constants, we have

$$a = a_0 N; \quad b = b_0 N\,. \tag{15}$$

The quantity B is essentially proportional to the price of labor. As N increases, when $S_u = 0$, the total amount of natural resources per capita decreases, and the amount of labor required to produce the same amount of goods increases also. Therefore the cost of production, and hence both A and B, increase with N. To a large extent this is counteracted by technological improvements. But no matter how great those technological improvements, for a constant S, when N becomes infinite, $B(N)$ must also become infinite. The coefficient $A(N)$ does not necessarily need to become infinite for $N = \infty$. The quantity K, as defined by G. Evans,[1] is also likely to increase with N. We thus have

$$n \leqq \frac{A(N)}{K(N)}[b_0 - a_0 B(N)]^2 N^2. \tag{16}$$

Since $B(\infty) = \infty$, therefore, for a finite value N^* of N, n will have to be zero. The value N^* is found as the root of the equation $a_0 B(N) = b_0$.

With increasing N, or density of population, private competition

under the principle of maximizing the individual profits eventually becomes impossible.

The only other alternative under the principle of maximizing individual profits is to form a monopoly or trust, dividing the total profit between all members. The profit in that case is (Reference 1, page 6):

$$\pi = \frac{[b_0 - B(N)a_0]^2 N^2}{4a_0[1 + a_0 NA(N)]N} - C(N). \tag{17}$$

With increasing N, regardless of the form $C(N)$, π becomes zero for a value $N_1^* < N^*$ of N. This will make the economic system, based on profit, impossible. The first term of the right side of equation (17) is always positive and tends for $N = \infty$ to

$$\frac{B(\infty)}{4A(\infty)}, \tag{18}$$

and the whole right side tends to

$$\frac{B(\infty)}{4A(\infty)} - C(\infty). \tag{19}$$

If the total profit π remains finite, then, since N_2 increases as N, the individual share of profit for every member of class II will decrease. It will decrease even if π becomes infinite, as long as

$$\frac{B(N)}{4A(N)} - C(N)$$

increases with N less rapidly than N. Under either of these conditions, the situation eventually arises where the profits per individual become too small, and the only way out of the difficulty is to keep the size of class II limited, by both preventing acceptance into class II of individuals II', and by *expelling* from class II individuals III', namely those of type III born in class II. But then a situation like the one described above will arise; namely, group II' will become strong enough to seize control of the situation and will introduce a behavior based on the denial of property rights to land.

It appears that no matter how we look at it, for $t > t_0$ the "capitalistic" system, based on maximizing the profit of the "organizing" class, will not lead to a situation in which everyone benefits, as is the case for $t < t_0$, $S_u > 0$.

The passive individuals of type III in class III, left alone, would be satisfied with the existing arrangements with class II. But the active individuals II' in class III, not having the opportunity to do

the same organizing work as the individuals *II* in class *II*, will organize the passives of class *III* and influence them to stand against class *II*, which sometimes results in advantages both to class *III* and to individuals *II'*. This may be the origin of organized labor movements. The intensity of the movements, as measured for instance by the membership in different unions or by the number of unions, is again given by an equation in the form of equation (12). A mathematical theory of labor movements can thus be developed and checked against available statistical observations.

The whole situation may be viewed from a still different angle. Consider the case, discussed in chapter v, pages 47–48. We found then that

$$\theta_m'' > \theta_m' > \theta_m . \tag{20}$$

Moreover, we found that in order for the society to exist, we must have

$$\theta w_2 w_3 > -\frac{\varepsilon_3}{f(\eta)}, \tag{21}$$

where w_2, w_3 and ε_3 stand for w_1, w_2 and ε_2 of chapter v. For $\varepsilon_3 > 0$ inequality (21) is always satisfied. With increasing population density, however, ε_3 is bound to become negative and to increase in absolute value. Inequality (21) may then be written

$$\theta > \frac{|\varepsilon_3|}{w_2 w_3 f(\eta)} . \tag{22}$$

The quantities w_2 and w_3 also increase with N, but they have upper limits w_{0_2} and w_{0_3}. Hence with increasing N, θ must increase eventually, and for some value of N the value of θ which satisfies inequality (22) will be greater than θ_m, which maximizes the rate of increase of wealth of class *II*. It will eventually come close enough to θ_m', the value which maximizes the total rate of increase of wealth of the whole society. Things will be happening as if, instead of using the principle of maximizing profit for the organizing class, the total welfare of the society were maximized.

If θ increases further and becomes near or equal to θ_m', then a situation may arise, where individuals of class *III* have a greater profit than those of class *II*, though class *II* supervises their work. Under these circumstances individuals of class *II* are likely to pass voluntarily into class *III*, and the whole system loses its organization. At the present stage of the theory, it is impossible to predict what will happen under those circumstances.

In the preceding chapter we remarked that developments of modern complex industries are likely to create a condition under which a "collectivistic" behavior is more satisfactory. In this chapter we find that a transition to "collectivistic" behavior is determined by the vanishing of S_u. Since S_u vanishes the sooner, the more rapidly s_{02} and s_0 grow with time, and since the increase of the latter is directly connected with engineering and industrial developments, therefore we again conclude that engineering developments create conditions which bring forth a change from "individualistic" to "collectivistic" behavior.

This also throws some light on the question why various socialist doctrines have become so prominent in the last few decades, although the differences between working classes and organizing classes existed for milleniums. The condition $S_u = 0$ may be becoming satisfied practically throughout the whole world. In previous centuries $S_u > 0$ was satisfied in most of the world.

REFERENCES

1. G. E. Evans, *Mathematical Introduction to Economics,* New York: McGraw-Hill, 1930.

CHAPTER XXI

SOME CONSIDERATIONS OF THE HISTORY OF A FEW NATIONS

In principle, the considerations of the preceding chapters should enable us to describe the general course of history of a nation, provided the initial conditions at a given time and the types of interactions occurring in the population are prescribed. We have seen in chapter xiv that with a constant profile of the society, but with limited interclass mobility, the duration of a given type of social structure would be of the order of few hundred years. It is interesting to note that this order of magnitude actually occurs in history. The Roman Empire lasted about 500 years, the French monarchy about 1,000 years, the Russian monarchy about 400 years.

The variation in profile of a society which affects both the composition of the total active group and its relative size, introduces complications but does not change fundamentally the orders of magnitude involved. The type of interaction taken as the basis of discussion in section III of chapter xiv seems to be the most likely. As has been pointed out on page 123, such a type of interaction may or may not lead to discontinuities in the social structure.

Some general considerations may be indicated here in regard to what determines the initial conditions of a society.

We have considered hitherto principally two types of active individuals: the military-administrative and the organizing. We pointed out, however, that actually there are many other types present (chapter vi, page 53). In particular, the class representing various religious and philosophic activities, in their broadest sense, may be very pronounced. Denoting the number of individuals in that class by z_0, we may have a situation where $z_0/(x_0 + y_0 + z_0)$ is almost equal to 1, while $x_0/(x_0 + y_0 + z_0) << 1$ and $y_0/(x_0 + y_0 + x_0) << 1$. As was remarked before (page 53), this may be representative of such countries as India and China.

We may also have to consider the class of active individuals in the field of pure arts. Let us denote their number by u_0. A country may have a rather large relative value of u_0/N and yet the absolute value of $(x_0 + y_0)/N$ may be small. The pre-revolutionary Russia apparently did not have a high value of y_0/N, as indicated by the

relatively small number of discoveries and inventions (cf. chapter xvii, Figure 2). But if we consider the per capita production of Russia in the field of music, literature, or painting, we find a figure well comparable to that of any other country. A country deficient in industrial and military organization may rank high in art.

There is another important point contrasting art activities with military, industrial, and even scientific, activities. A general without an army or an executive without employees are unreal abstractions. An inventor without a well organized industry to use and promote his inventions is also of no more than a limited value. All those active people need a sufficient number of passive ones in order to enable their activities to survive. Even a purely scientific discovery made by a great man is developed and improved by many lesser scientists. This is not so with art. Here the activity is attached practically to one individual. Passive ones are not essential for artistic creations to last. Few scientists now read Newton or Huygens in the original, their creations having been digested and developed by generations of scientists. But we still listen to Bach's or Beethoven's music essentially as it was written by Bach or Beethoven. Thus the artistic activities of a social group are governed by relations different from those governing other social behavior.

The conditions which determine the sizes and compositions of active groups in different nations could theoretically be traced to the beginning of humanity, and present essentially a biological problem. As humanity originated somewhere on the Central Asiatic Plateau and began to spread, those social groups which had large values of x_0/N and y_0/N were more likely to move away to a greater distance in search of either conquests or more earthly goods than those groups characterized by a large z_0/N and small x_0/N and y_0/N. We would therefore expect the nations in the immediate neighborhood of the "cradle of humanity" to have a small value of $(x_0 + y_0)/N$ and a large value of z_0/N. The above-mentioned cases of China and India fit into this picture.

If the initial migration goes into uninhabited places and is offered relatively little resistance, we may expect that nations with large $y_0/(x_0 + y_0)$'s will be involved. These are led by active individuals who are primarily interested in improvement of the conditions of life and in accumulation of wealth. On the other hand, subsequent migrations, which are connected with conquests, will involve nations with a large $s = x_0/(x_0 + y_0)$. It is possible that the prehistoric migrations into Europe were of the first type, those of the 4th and 5th Centuries A.D. of the second. It would be of interest to develop an abstract theory of the phenomenon and thus obtain the composition of the active

class of each nation as a function of the distance it had to travel and of the time when the migration occurred.

The westermost nations in Europe would therefore be expected to have had originally a rather large value of $s_1 = y_0/(x_0 + y_0)$, while in the East larger values of $x_0/(x_0 + y_0)$ would prevail.

It is of interest to observe that the early Russians in the 8th-12th Century A.D. were essentially trades people. There was not much of the purely Russian military, the military functions being performed by the Norsemen, who originally formed a sort of hired army. The general setup of the social life at that time was what would be termed "democratic" now; the controlling class was the industrial and trade class.

From the 12th Century the situation changes, the social profile showing an increased ratio of $x_0/(x_0 + y_0)$. This may have been due to several causes. First, there might have been a change of a nature discussed in chapter xv. Then, the infusion of Tartar blood with the large value of $x_0/(x_0 + y_0)$ in the population would have the same effect. The constant fighting may have produced conditions more favorable to the survival of the military class.

From about the 13th or 14th Century the history of Russia may in general be described on the basis of a constant composition of the active group with a large value of s. We originally have the closed ruling class of the princes, allegedly of Norse descent, which eventually thins out, approximately as described in chapter xiv, section I, and has to cede control to active individuals of the same type who are born of the passive class. This transition takes place at the time of John the Terrible, 16th Century, and is, in a way, characterized by a drastic social revolution. The new ruling group again forms a closed class, that of the landed nobility, which lasts until 1917, to give way again in the manner described in chapter xiv, section I, to rulers born of the passive class, who are in *some* respects, however, of a similar type. In spite of a drastic change of ideology and social structure in the directions discussed in chapters xix and xx, the regime in Soviet Russia is *in its methods* reminiscent of the Imperial regime.

France, as we should expect, begins with a large s in the Middle Ages, the value of x_0/N being large enough, so that the thinning out process takes a millenium. However, if a sufficiently large industrial class is present, then when the military class is thinned out, the following situation may arise:

Using the notations of section I of chapter xiv, we may have two situations: either $n_{II}^A > n_B^B$, or $n_{II}^A < n_B^B$. The symbol A now refers to the military class; B, to the industrial. In the former case, if the inequality is sufficiently strong, n_{II}^A gains control. In the sec-

ond case $n_B{}^B$, the industrial class, takes control since the individuals of type A are divided against themselves—part belonging to class I, part to class II. But although $n_{II}{}^A < n_B{}^B$, the total number of individuals A remains greater than $n_B{}^B$ for a constant profile; thus $n_I{}^A + n_{II}{}^A > n_B{}^B$. Therefore, eventually, after a sufficient lapse of time, type A again gains control. The time necessary for this can be estimated from considerations of chapter xiv, section I, to be of the order of 100 years.

In England there seems to have been initially (1066) a fair balance between x_0 and y_0, with a preponderance of the former, and a gradual shift towards the present preponderance of y_0. It seems to be the case of a variable composition of the active group, in other words, of a variable s.

The history of the United States presents a unique feature. That country was created by emigrés who revolted against the autocratic regimes of Europe and therefore represented a selection of people of the organizing and industrial class. Hence, we have the American democracy from the beginning. Due to the difference between parents and progeny, the active industrial class in the United States gradually "thins out," and a shift towards smaller values of $y_0/(x_0 + y_0)$ occurs. Various governmental controls introduced since 1932 may be considered as a consequence of it. To what extent this shift towards governmental control will continue cannot be predicted on the basis of the present theory.

The unprecendented depression of 1932 may well be a case of loss of control by the "thinned out" industrial class, of the type discussed in chapter xviii.

History is difficult to describe mathematically because we have hardly any data of a quantitative nature. There is, however, one set of data that are definitely quantitative; these are the chronological points of sudden social changes or revolutions and wars. We might represent the course of historic events in a country by equations developed in previous chapters, choosing the parameters so as to give the chronology of the discontinuous changes correctly. Due to the oversimplifications used in the theory, this would as yet have little meaning. A further improvement of the theory seems to be indicated along the lines of chapter xvii by considering not one active group but two or more. Considering two active groups, a military and an industrial one, we should put the death rates as functions of the available natural resources on one hand, and of the ratio y_0/N on the other hand. We might in this way find general changes in profile which, by varying the parameters, could be applied to all nations. One could also develop a theory of the connection between considerations

of chapter iii and those of chapter vi, section II. The one deals with control of the passive population, the other with mutual relations of classes.

CHAPTER XXII

THE DYNAMICS OF PHYSICAL CONFLICT BETWEEN SOCIAL GROUPS

In previous chapters we discussed the relations which govern the influence of a relatively small group of "leading" individuals upon the behavior of the larger number of other individuals. These considerations may be applied at first *in abstracto* to a rather interesting and important social problem, namely, to the physical conflict between two large groups of individuals, each being led by a corresponding smaller group. A large number of social phenomena, ranging from street riots to wars involving two or several nations, correspond to the abstract picture which we shall discuss here.

Consider two populations, one (population 1) composed of M, the other (population 2), of N individuals. Let most of the individuals belong to type II, which we shall again call the "passive type." Let x_M individuals in the first and x_N individuals in the second population be of type I, the "active type." Suppose that the active groups of the two populations are in mutual conflict and that they endeavor to influence the other individuals in their populations in such a way that they would participate in the conflict. In general, each population may have two groups of active individuals of opposite behavior; a number x, for instance, wishing a conflict and a number y opposing it. The behavior of the whole population will then be determined by the considerations of chapter iii which led to expressions (8) and (9) of that chapter. For simplicity, we shall consider here the case in which there is no opposing group. We may then use expressions (8) in which we make $y_0 = 0$, and for x_0 and N of that expression we may substitute x_M and M or, correspondingly, x_N and N. Thus both populations will participate as a whole in a conflict, if

$$x_M > \frac{a_1}{a_{01} + a_1} M; \quad x_N > \frac{a_2}{a_{02} + a_2} N. \tag{1}$$

The coefficients a_1 and a_{01} and, correspondingly, a_2 and a_{02} have the same meaning as a and a_0 of chapter iii. The subscripts are added to take into account possible differences in the constants for the two populations.

Putting

$$\frac{a_1}{a_{01}+a_1}=r_1; \quad \frac{a_2}{a_{02}+a_2}=r_2; \tag{2}$$

we may write inequality (1) thus:

$$x_M > r_1 M \; ; \quad x_N > r_2 N \, . \tag{3}$$

Let the conflict consist of an actual destruction of the individuals of one population by those of the other. The same considerations, however, will apply to the case where, instead of an actual destruction, we have just a temporary elimination by putting the individuals "hors de combat." For brevity we shall use the word "destroy."

Let each individual of population 1 destroy k_1^2 individuals of population 2, while each individual of population 2 destroys k_2^2 individuals of population 1. We then have

$$\begin{aligned} \frac{dM}{dt} &= - k_2^2 N ; \\ \frac{dN}{dt} &= - k_1^2 M . \end{aligned} \tag{4}$$

We may consider the more general case in which k_1^2 and k_2^2 are different for the active and passive individuals. We shall limit ourselves first to the more restricted case, however.

Equations (4) integrated by the standard method give:

$$\begin{aligned} M &= \frac{1}{2}\left(M_0 - \frac{k_2}{k_1}N_0\right)e^{k^2t} + \frac{1}{2}\left(M_0 + \frac{k_2}{k_1}N_0\right)e^{-k^2t} ; \\ N &= -\frac{1}{2}\left(M_0 - \frac{k_2}{k_1}N_0\right)\frac{k_1}{k_2}e^{k^2t} + \frac{1}{2}\left(M_0 + \frac{k_2}{k_1}N_0\right)\frac{k_1}{k_2}e^{-k^2t} ; \end{aligned} \tag{5}$$

$$k^2 = \sqrt{k_1^2 k_2^2} = + k_1 k_2 \, . \tag{6}$$

The quantities N_0 and M_0 are the total number of individuals for $t = 0$.

If $M_0 > (k_2/k_1)N_0$, then M consists of two positive terms, one increasing, the other decreasing. For $t = 0$, $M = M_0$, the first term increases, the second decreases. The increase is, however, less rapid than the decrease. Therefore M will first decrease, then increase. But from the first equation (4) it follows that when M reaches a minimum, so that $dM/dt = 0$, we also have $N = 0$. Thus at the moment $t = t_2^*$, at which M has a minimum, $N = 0$. This means that at $t = t_2^*$, the second population becomes completely destroyed and the

conflict ceases through a victory of the first. After that moment there is no further conflict, so that M does not actually increase.

If $M_0 < (k_2/k_1)N_0$, then M becomes zero, while N reaches a minimum.

While the leaders of the two populations will in general participate in the active conflict, they may be exposed to the danger of destruction to a different extent than the passive individuals. If their exposure to danger is the same as that of the passive individuals, we have:

$$\frac{dx_M}{dt} = \frac{x_M}{M}\frac{dM}{dt},$$
$$\frac{dx_N}{dt} = \frac{x_N}{N}\frac{dN}{dt}, \tag{7}$$

which integrated gives

$$\frac{x_M}{M} = \text{Const.}; \quad \frac{x_N}{N} = \text{Const.} \tag{8}$$

Under these conditions, if inequalities (3) were satisfied at the beginning, they will remain satisfied throughout the conflict and the conflict will therefore continue until one of the populations is completely destroyed, that is, until either $t = t_1^*$ or $t = t_2^*$. These values are obtained respectively as the roots of the equations $M = 0$ and $N = 0$, into which we substitute expressions (5) and which we solve for t. If $M_0 < \frac{k_2}{k_1} N_0$, then t_2^* is imaginary, while

$$t_1^* = \frac{1}{2k^2} \log \frac{\frac{k_2}{k_1} N_0 + M_0}{\frac{k_2}{k_1} N_0 - M_0}. \tag{9}$$

If $M_0 > \frac{k_2}{k_1} N_0$, then t_1^* is imaginary and

$$t_2^* = \frac{1}{2k^2} \log \frac{M_0 + \frac{k_2}{k_1} N_0}{M_0 - \frac{k_2}{k_1} N_0}. \tag{10}$$

In general, we shall have instead of equations (7):

$$\frac{dx_M}{dt}=\left(\frac{x_M}{M}\right)^m\frac{dM}{dt} \tag{11}$$

and

$$\frac{dx_N}{dt}=\left(\frac{x_N}{N}\right)^n\frac{dN}{dt}, \tag{12}$$

where m and n may be either smaller or greater than one.

Equations (11) and (12) give:

$$\frac{dx_M}{x_M{}^m}=\frac{dM}{M^m} \tag{13}$$

and

$$\frac{dx_N}{x_N{}^n}=\frac{dN}{N^n}. \tag{14}$$

Equations (13) and (14) give:

$$\frac{1}{x_M{}^{m-1}}=\frac{1}{M^{m-1}}+C_1 \tag{15}$$

and

$$\frac{1}{x_N{}^{n-1}}=\frac{1}{N^{n-1}}+C_2. \tag{16}$$

C_1 and C_2 are integration constants determined by the requirements that for $M=M_0$, $x_M=x_{0M}$ and for $N=N_0$, $x_N=x_{0N}$, where x_{0M} and x_{0N} are the initial values of x_M and x_N. This gives:

$$C_1=\frac{M_0{}^{m-1}-x_{0M}{}^{m-1}}{M_0{}^{m-1}\,x_{0M}{}^{m-1}};\quad C_2=\frac{N_0{}^{n-1}-x_{0N}{}^{n-1}}{N_0{}^{n-1}\,x_{0N}{}^{n-1}}. \tag{17}$$

For $m<1$ and $n<1$, $C_1<0$ and $C_2<0$. For $m>1$ and $n>1$, $C_1>0$ and $C_2>0$. Equations (15) and (16) give:

$$\frac{x_M}{M}=\sqrt[m-1]{\frac{1}{1+C_1M^{m-1}}},\quad \frac{x_N}{N}=\sqrt[n-1]{\frac{1}{1+C_2N^{n-1}}}. \tag{18}$$

For $m<1$, x_M/M decreases with decreasing M, because in that case $C_1<0$. The same thing holds for x_N/N. If $n<1$, the destruction of the active individuals in the first population goes on more rapidly than that of the passive. As a result, x_M/M decreases and if it becomes so small that $x_M<r_1M$, while for the other population we still have $x_N>r_2N$, then the active group can no more influence the passive individuals and make them continue the fight. The population stops fighting, becomes demoralized, although at this moment we may

still have $M \geqq N$. Similar considerations hold for the second population. On the other hand, if $m > 1$ and $n > 1$, then x_M/M and x_N/N increase and the fight continues until one of the populations is completely destroyed.

We thus have a quantitative interpretation for the "breakdown of morale," which is usually a rather elusive notion.

For $m < 1$, the "breakdown" of the first population occurs at the moment t_1 at which

$$\frac{x_M}{M} = r_1, \tag{19}$$

which, because of equations (18), gives:

$$M = \sqrt[m-1]{\frac{r_1^{1-m} - 1}{C_1}}. \tag{20}$$

Introducing into equation (20) the expression for M from expressions (5) and solving with respect to t, we thus find the moment t_1. In a similar way, we find the moment t_2 at which the second population would break down if the conflict continued. Depending on whether $t_1 < t_2$ or $t_1 > t_2$, the first or the second population will be defeated in the conflict.

Such considerations as the above cannot of course apply to such complex phenomena as a war involving several nations at once. Some general considerations on that subject may, however, be of interest.

CHAPTER XXIII

OUTLINE OF A MATHEMATICAL THEORY OF WAR

In a previous chapter we outlined a theory of physical conflict between two social groups, both consisting of "active" and "passive" individuals. Such a physical conflict, as we have seen, may either end in a complete physical destruction of one of the groups or else in a "breakdown of morale" of one of the groups, even though its physical strength still remains appreciable. This "breakdown of morale" occurs if and when, due to the differential rates of destruction of active and and passive individuals, the relative number of the former drops below the critical value necessary for the control, for the leading, of the latter.

Those considerations were confined to purely temporal relations, neglecting possible shifts in space of the fighting groups. In reality such shifts in space form a very important part of any physical conflict, be it a street riot or a war, but especially in the latter. We shall therefore attempt to take these shifts into consideration.

Denoting by m and n the numbers of individuals in the two conflicting populations, we have for the rates of destruction of the populations in absence of spatial shifts equations (4) of chapter xxii:

$$\frac{dm}{dt} = -k_2 n, \quad \frac{dn}{dt} = -k_1 m, \tag{1}$$

with

$$k_1 > 0 \quad \text{and} \quad k_2 > 0 . \tag{2}$$

In actual warfare the combatant groups do not usually remain on the same spot, but one of them sooner or later begins to retreat. The causes determining the retreat are manifold, frequently of a psychological nature. We may consider here as one of the possibly important causes the relative rate of destruction of the group, $(1/n)(dn/dt)$ or $(1/m)(dm/dt)$. Assuming that at least at the beginning of the conflict there is a sufficient excess of actives over the required minimum threshold values, the psychological factors in retreat will be secondary. But for purely physical considerations, when the relative rate of destruction of a group exceeds a certain fraction α, $(0 < \alpha < 1)$, the only way to decrease this rate of de-

struction may be to put temporarily a sufficiently safe distance between that group and the enemy. Suppose that

$$\left|\frac{1}{m}\frac{dm}{dt}\right| > \left|\frac{1}{n}\frac{dn}{dt}\right| . \tag{3}$$

As long as the absolute values of both expressions are less than α, the fight will continue on the same place. But if

$$\left|\frac{1}{m}\frac{dm}{dt}\right| > \alpha, \quad \left|\frac{1}{n}\frac{dn}{dt}\right| < \alpha, \tag{4}$$

then the first group will retreat.

The faster it retreats, the fewer will be its losses. If we denote the speed of retreat by v_1, then the losses will be decreasing with increasing v_1. When v_1 reaches a value v_{01}' at which the enemy cannot catch up with the retreating group, the losses will be zero. As group *I* retreats, group *II* will follow; but the retreat of group *I* in general lessens the losses it can inflict on group *II*. Thus the relative losses of group *II* will also decrease with increasing v_1, but in general according to a different law. Within a sufficiently small range we may put the losses of group *I* proportional to $(v_{01}' - v_1)$, and those of group *II* proportional to $(v_{01}'' - v_1)$, where v_{01}'' is another constant.

Relations (1) and (3) give

$$\frac{k_2 n}{m} > \frac{k_1 m}{n} \quad \text{or} \quad k_2 n^2 > k_1 m^2 \tag{5}$$

Hence if m and n are approximately equal, the group with a smaller k will retreat.

Inasmuch as it usually is in the interest of both fighting groups to possess as much territory as possible, the retreating group will adjust its velocity of retreat v to the possible minimum, namely, it will make v_1 just large enough to keep the relative losses just below α.

Instead of equations (1) we now have, denoting by β_1 and β_2 two coefficients,

$$\begin{aligned} \frac{dm}{dt} &= -k_2\beta_1(v_{01}' - v_1)n; \\ \frac{dn}{dt} &= -k_1\beta_2(v_{01}'' - v_1)m; \\ \frac{1}{m}\frac{dm}{dt} &= -\alpha. \end{aligned} \tag{6}$$

If k_1, k_2 and α remain constant throughout the whole process of conflict, then the third equation gives us m directly as a function of time. By substituting that function into the first equation, solving it for n, and differentiating with respect to t, we obtain, together with the second equation, a system of two non-linear first order simultaneous equations for the determination of n and v_1 as functions of time.

Actually it is more likely that k_1, k_2, and α themselves are varying during the process of war. The more territory that remains behind the retreating army, or the more there is to fall back on, the greater the tendency to spare the fighting strength, and the smaller the α. Hence α will in general be an increasing function of the distance retreated, or

$$\alpha = F\left(\int_0^t v_1\, dt\right). \tag{7}$$

The coefficients k_1 and k_2 will also vary in general. They depend primarily on the technical equipment of the two fighting groups, such as the amount and quality of ammunition, fighting machinery, transportation, etc. In other words, they depend on the industrial productivities of the two fighting countries. These in turn are the greater, the greater the natural resources and therefore the larger the area of the territory held. Both sides will tend to increase their k, and the simplest assumption which we can make is that within certain limits the rate of increase of k's is proportional to the territory held. Of course eventually the values of k should reach saturation values. We thus have approximately, denoting by S_{01} and S_{02} the areas of the territory held initially by the two belligerents,

$$\frac{dk_1}{dt} = a_1\left(S_{01} - b_1 \int_0^t v_1\, dt\right); \tag{8}$$

$$\frac{dk_2}{dt} = a_2\left(S_{02} + b_2 \int_0^t v_1\, dt\right). \tag{9}$$

The quantities a_1, b_1, a_2 and b_2 are constant coefficients. Of those b_1 and b_2 are largely determined by the geometrical shape of the territories involved.

The more complete system of equations which govern the process of fighting is now:

$$\frac{dm(t)}{dt} = -\beta_1 k_2(t)\,[v_{01}' - v_1(t)]\, n(t); \tag{10}$$

$$\frac{dn(t)}{dt} = -\beta_2 k_1(t)\,[v_{01}'' - v_1(t)]\,m(t)\,; \tag{11}$$

$$\frac{1}{m(t)}\frac{dm(t)}{dt} = -F\left(\int_0^t v_1(t)\,dt\right)\;; \tag{12}$$

and equations (8) and (9). These non-linear integrodifferential equations determine the five functions $n(t)$, $m(t)$, $v_1(t)$, $k_1(t)$, and $k_2(t)$.

Even if we solve the system of equations (8)–(12), we shall not determine completely the course and outcome of the conflict. Two more factors have to be considered.

The initial conditions which determine the solution of the system (8)–(12) are supposed to be such that inequalities (5) hold and that group I retreats. If the solution of the system (8)–(12) is such that in the course of time the inequality (5) becomes reversed, then from that time t^* on, the second group begins to retreat. This does not mean a mere change in sign of v_1, but an actual discontinuity in the system of solutions. For $t > t^*$ we have a similar system of equations, in which for v_1, v_{01}', and v_{01}'' we must substitute v_2, v_{02}', and v_{02}'', and use as initial condition the values $m(t^*)$, $n(t^*)$, $k_1(t^*)$, and $k_2(t^*)$. The movement now proceeds in the opposite direction. Another reversal may in principle occur again at $t_1^* > t^*$, and so on.

We may try to avoid this rather inelegant situation by complicating somewhat further our system of equations. We may put, with c as a coefficient,

$$v = c(k_2 n^2 - k_1 m^2), \tag{13}$$

a positive v meaning the retreat of group I. We now may put, for instance,

$$\begin{aligned}
\frac{dm(t)}{dt} &= -\beta_1 k_2(t)\{v_{01}' - c\,[k_2(t)n^2(t) - k_1(t)m^2(t)]\}\,;\\
\frac{dn(t)}{dt} &= -\beta_2 k_1(t)\{v_{01}'' - c\,[k_2(t)n^2(t) - k_1(t)m^2(t)]\}\,;\\
\frac{dk_1(t)}{dt} &= a_1\{S_{01} - b_1 c\int_0^t [(k_2(t)n^2(t) - k_1(t)m^2(t)]dt\}\,;\\
\frac{dk_2(t)}{dt} &= a_2\{S_{02} - b_2 c\int_0^t [k_2(t)n^2(t) - k_1(t)m^2(t)]dt\}.
\end{aligned} \tag{14}$$

Equation (12) is now superfluous since the speed of retreat is determined by (13).

The second factor to be considered is the change in the ratios of

actives to passives in both groups. Denote the number of active individuals in group *I* by x_1, in group *II* by x_2. Denote correspondingly the number of passives by y_1 and y_2, so that

$$x_1 + y_1 = m; \quad x_2 + y_2 = n. \tag{15}$$

Since usually $x_1 << y_1$, $x_2 << y_2$, therefore approximately

$$y_1 = m; \quad y_2 = n. \tag{16}$$

The relative losses of the actives and passives will in general be different. Hence

$$\frac{dx_1/dt}{dy_1/dt} = \gamma_1 \frac{x_1}{y_1}; \quad \frac{dx_2/dt}{dy_2/dt} = \gamma_2 \frac{x_2}{y_2}; \tag{17}$$

where γ_1 and γ_2 are in general different from 1. Denoting by x_{01}, y_{01}, x_{02}, and y_{02} the initial values of x_1, y_1, x_2, and y_2, we have from equation (17):

$$\frac{x_1}{y_1^{\gamma_1}} = \frac{x_{01}}{y_{01}^{\gamma_1}} = B_1; \quad \frac{x_2}{y_2^{\gamma_2}} = \frac{x_{02}}{y_{02}^{\gamma_2}} = B_2, \tag{18}$$

or because of (16):

$$\frac{x_1}{y_1} = B_1 m^{\gamma_1 - 1}; \quad \frac{x_2}{y_2} = B_2 n^{\gamma_2 - 1}. \tag{19}$$

In the course of the conflict, m and n decrease. If, as is likely to be the case, $\gamma_1 > 1$ and $\gamma_2 > 1$, then x_1/y_1 and x_2/y_2 decrease even more rapidly than m and n. The system of equations (8)-(12), or the alternate system (14), gives us m and n as functions of time. Equations (19) then give us x_1/y_1 and x_2/y_2 as functions of time. If at any time t, either x_1/y_1 or x_2/y_2 drops below the threshold h_1 or h_2, determined by the considerations of chapters iii and xviii and necessary to maintain the influence of the actives over the passives, then that group for which this occurs ceases fighting even though physically able to continue. Thus it may occur that group *I* will begin to throw group *II* back, yet eventually break down before group *II* is completely expelled from the territory of group *I*.

Equations (8)-(12) or (14) together with (19) determine if and when this will happen. If the solution of equations (8)-(12) or (14) is such that both x_1/y_1 and x_2/y_2 never drop below their corresponding thresholds, the conflict ends when either m or n becomes zero.

Any attempt at solving the system (8)-(12) or (14) with even a remote exactness would be very difficult. As an illustration of what can in principle be obtained from such a mathematical approach, we shall consider the simple case of equations (6), and even this we shall treat

only approximately and under some very crude oversimplifying assumptions.

Let us consider the following situation. Group *II* attacks group *I*, having beforehand provided for a maximum possible k_2, so that inequalities (5) are satisfied, and group *I* retreats, k_2 remaining constant. But k_1 will increase according to equations (8). However, we shall make the rather artificial assumption that the phenomenon of retreat continues in such a way *as if* k_1 remained constant, equal to its initial value k_{01}. Only when k_1 has increased sufficiently to reverse inequality (5) will its full value be used. While this assumption is in a way artificial, it is psychologically plausible. The constant k_1 determines the striking power of group *I*, not its power of defense. An army may well retreat without much attempt to slow down or to hit back until it has accumulated all the necessary striking power.

Furthermore, we shall assume α to be constant and

$$\beta_1 = \beta_2 = \beta ; \quad v_{01}' = v_{01}'' = v_0 .$$

With the above assumptions the first two equations (6) give

$$\frac{dn}{dm} = k\frac{m}{n} ; \quad k = \frac{k_1}{k_2} , \tag{20}$$

since we assumed that until k_1 becomes sufficiently large, things happen as if it had a constant value. Equation (20) gives

$$n^2 = km^2 + C , \tag{21}$$

where C is determined from the requirement that when m has its initial value m_0, n has the initial value n_0. This gives:

$$n = \sqrt{km^2 + n_0^2 - km_0^2} . \tag{22}$$

The third equation (6) gives

$$m = m_0 \, e^{-\alpha t} , \tag{23}$$

while the first and the third give

$$k_2 \beta (v_0 - v_1) n = \alpha m . \tag{24}$$

Equations (22), (23), and (24) now give

$$v_1 = v_0 - \frac{\alpha \, m_0 \, e^{-\alpha t}}{k_2 \beta \sqrt{km_0^2 \, e^{-2\alpha t} + n_0^2 - km_0^2}} . \tag{25}$$

For sufficiently large values of t, when α is not too small, equation (25) may be written approximately thus:

$$v_1 = v_0 - \gamma e^{-\alpha t}; \quad \gamma = \frac{\alpha\, m_0}{k_2 \beta \sqrt{n_0^2 - k m_0^2}}. \tag{26}$$

The coefficient γ is real because of inequalities (5).

While the conflict goes on as if k_1 were constant, the "potential" value of k_1 increases according to equation (8). Hence, from expressions (26),

$$\frac{dk_1}{dt} = a_1 [S_{01} - b_1 v_0 t + \frac{b_1 \gamma}{\alpha}(1 - e^{-\alpha t})], \tag{27}$$

or, integrating,

$$k_1 = k_{01} + a_1 \left(S_{01} + \frac{b_1 \gamma}{\alpha}\right) t - \frac{a_1 b_1 v_0}{2} t^2 - \frac{a_1 b_1 \gamma}{\alpha^2}(1 - e^{-\alpha t}). \tag{28}$$

When the "potential" value of k_1, as given by equation (28), becomes so large that upon substitution into inequalities (5) it reverses that inequality, then group *I* takes the offensive. To compute the "potential" value which k_1 must reach to that effect, we must substitute an equality sign into (5), and solve it for k_1. This gives

$$k_1 = \frac{k_2\, n^2}{m^2}. \tag{29}$$

Because of expressions (22) and (20), equation (29) may be written:

$$k_1 = k_1 + \frac{k_2 n_0^2}{m^2} - \frac{k_1 m_0^2}{m^2}, \tag{30}$$

or

$$k_1 = \frac{k_2 n_0^2}{m_0^2}. \tag{31}$$

The time t^* is determined by equating (28) and (31) and solving for t. To do this approximately, we expand the exponential in expression (28) and break off at the quadratic terms. The equation for the determination of t^* then becomes

$$\frac{a_1 b_1}{2}(v_0 - \gamma)\, t^2 - a_1 S_{01} t + \frac{k_2 n_0^2}{m_0^2} - k_{01} = 0. \tag{32}$$

We always have $v_1 < v_0$, and this holds also for $t = 0$. Hence, because of equations (26):

$$v_0 - \gamma > 0. \tag{33}$$

In order for equation (32) to have any real roots at all, we must have

$$a_1^2 S_{01}^2 - 2a_1 b_1 (v_0 - \gamma) \left(\frac{k_2 n_0^2}{m_0^2} - k_{01} \right) \geqq 0. \tag{34}$$

If inequality (34) holds, then equation (32) has either only a single positive root, or two positive roots. In the latter case the smaller one must be chosen, since inequalities (5) hold for $t = 0$. Thus we find

$$t^* = \frac{a_1 S_{01} - \sqrt{a_1^2 S_{01}^2 - 2a_1 b_1 (v_0 - \gamma) \left(\frac{k_2 n_0^2}{m_0^2} - k_{01} \right)}}{a_1 b_1 (v_0 - \gamma)}. \tag{35}$$

Condition (34) is necessary but not sufficient for the first group to take the offensive eventually. In order to have this actually happen, it is necessary that at the time t^* the ratio x_1/y_1 still remains above the necessary threshold h_1.

The first equation of (19) together with (23) gives, because of expressions (16) and (18)

$$\frac{x_1}{y_1} = \frac{x_{01}}{m_0} e^{-(\gamma_1 - 1) a t}. \tag{36}$$

Hence, in addition to inequality (34), we must have

$$\frac{x_{01}}{m_0} e^{-(\gamma_1 - 1) a t^*} > h_1. \tag{37}$$

If both inequalities (34) and (37) hold, then at t^* the first group begins the offensive. This in itself does not mean that it will eventually win the fight, for beginning with $t = t^*$, the struggle is described again by equations of the form (6), in which we now have v_2 instead of v_1, and as initial condition we have the values $m(t^*)$ and $n(t^*)$ of m and n at $t = t^*$. It may happen that during the course of the retreat of group *II*, x_1/y_1 will drop below h_1 sooner than x_2/y_2 will drop below h_2. Whether and when this happens can be determined by considerations rather similar to the ones above. Fluctuations back and forth are possible depending on the values of the different constants or group *II* may break down before group *I*.

But by applying the above reasoning, it is always possible for the simpler case, contemplated here, to predict the outcome of the struggle and its total duration if all the constants entering into equations (6) and (8), as well as the initial conditions, are given.

If the second term of the discriminant of (32) is much smaller than the first, then by factoring $a_1 S_{01}$, expanding the radical in expression (35), and retaining only the linear term, we find the expres-

sion for t^* simplifies to

$$t^* = \frac{\dfrac{k^2 n_0^2}{m_0^2} - k_{01}}{a_1 S_{01}}, \tag{38}$$

while inequality (37) becomes

$$\frac{\dfrac{k^2 n_0^2}{m_0^2} - k_{01}}{a_1 S_{01}} < \frac{1}{(\gamma_1 - 1)\alpha} \log \frac{x_{01}}{m_0 h_1}. \tag{39}$$

The conditions for an eventual offensive after an initial retreat are the better, the larger S_{01}, the larger m_0 and k_{01}, and the smaller γ_1 and α.

We may consider that m_0 and n_0 will depend largely on the total populations M and N of the conflicting countries, as well as on their population densities. More densely populated countries will have a larger tendency to expand and will keep a proportionally greater army before the struggle. Plausibly we may put

$$m_0 = \eta \frac{M}{S_{01}}, \quad n_0 = \eta \frac{N}{S_{02}}, \tag{40}$$

where η is a constant, the same for all countries. Or we may take a more complex case:

$$m_0 = M \left(\eta_1 + \eta_2 \frac{M}{S_{01}} \right), \quad n_0 = N \left(\eta_1 + \eta_2 \frac{N}{S_{02}} \right), \tag{41}$$

with two constants η_1 and η_2.

In this case the different conditions, such as (34) and (37) express necessary relations between such quantities as the total populations, total areas, and the relative size of the active groups, all quantities that have already been introduced into mathematical sociology.

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